

ON PARTIALLY HYPOELLIPTIC OPERATORS. PART II: PSEUDO DIFFERENTIAL OPERATORS

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ABSTRACT. This study is an attempt at generalizing the class of partially hypoelliptic differential operators to a class of pseudodifferential operators, Symbol ideals are formed on the set of lineality and we discuss suitable topologies that allow the generalizations.

1. INTRODUCTION

The results in this study are an attempt to generalize results on formally partially hypoelliptic differential operators (cf. [7]) to more general operators. For a discussion on representations of pseudo differential operators we refer to ([7]). For a discussion on fundamental results on the class of differential operators, see ([8]).

The generalization is based upon symbols in a (geometric) ideal $f \in (I) = (\ker h)$, and $f(\zeta) = F(\gamma)(\zeta)$, where γ is considered in a dynamical system (or pseudo base), F is a lifting operator and ζ in a domain of holomorphy. The micro-local analysis is based upon the set of lineality and we will discuss generalizations of this set of invariance. We consider for instance the weighted lineality, of points η , such that $\frac{Q(x+\eta)}{Q(x)}\tau_\eta h\tau_\eta = h$, for a self-adjoint and hypoelliptic polynomial Q and where τ_η denotes translation. A longer discussion on this set appears in the end of this study.

Allthrough the study we will use the concept of monotropy (cf. [6]). A monotropic function is a continuous function f such that there exists a holomorphic function with zero's within ϵ distance from the zero's to f . We consider monotropic ideals such that $h = \tau_\epsilon w$, where w is algebraic in the sense that $w(x^2) = w^2(x)$ and we write $h \sim_m w$. A longer discussion on the monotropic functions and functionals is given in ([7]).

2. SYMBOLS IN A GEOMETRIC IDEAL

Reduced operators. Let's define a real set, called the set of lineality, as the zero's to a holomorphic function $\varphi_\lambda(it\eta) = P_\lambda(\xi + it\eta) - P_\lambda(\xi)$,

$$\Delta_C = \{\eta \in \mathbf{R}^m; \quad P(\xi + it\eta) - P(\xi) = 0 \quad \forall \xi \in \mathbf{R}^\nu, \quad \forall t \in \mathbf{R}\} = Z_{\varphi_\lambda}$$

We can define a pseudoconvex neighborhood of Z_{φ_λ} using $(1 + c|\zeta|^k |\varphi_\lambda(\zeta)|^2 = Ch(\zeta)$ for a holomorphic function $h \in H(U')$, $U \subset U'$, for an open set U' , for positive constants c, C and k . Let $g(\zeta) = Ch(\zeta) - (1 + |\zeta|^k |\varphi_\lambda(\zeta)|^2)$, be the holomorphic function defining U . P_λ denotes for $\lambda \in \mathbf{C}$ the operator $P - \lambda$.

Lemma 2.1. *If P_λ is reduced for complex lineality (with respect to one dimensional translations), thus $\Delta_{\mathbf{C}} = \{0\}$, then there are positive constants c, C , such that $(1 + |\xi|)^c \leq C |P_\lambda(\xi)|$ for all $\xi \in \mathbf{R}^\nu$.*

Assume \mathcal{H}_c is the class of polynomials satisfying the inequality in Lemma 2.1, for some constant c . For $h(\zeta) = 0$, since g^t is reduced, for some positive integer t , there is a positive σ , such that $|\xi|^\sigma \leq |g(\xi)|$, for large $\xi \in \mathbf{R}^m$. Thus, $|\xi|^{t\sigma - k''} \leq C |\varphi_\lambda(\xi)|^{2t}$, for some positive k'' . For $|h(\zeta)| > 0$, the inequality for h , is immediate. Assuming the defining polynomial is self-adjoint, we conclude that $\varphi_\lambda^t \in \mathcal{H}_{\frac{1}{2}(t\sigma - k'')}(\mathbf{R}^m)$. This means, for a sufficient number of iterations, $\sigma > k''/t$ and the polynomial is reduced. The constant $k'' < 1$, so there is a positive integer N (possibly smaller than t), such that $\sigma > 1/N$ and φ_λ^N is hypoelliptic in L^2 . Note that this hypoellipticity is not dependent on λ .

Proposition 2.2. *If $P(D)$ is a constant coefficients, partially hypoelliptic operator, there is an iteration index N_0 , such that $P(D)^N$ is hypoelliptic in \mathcal{D}' , for all $N \geq N_0$.*

We can prove a similar result for self-adjointness. The notation $P \prec Q$ indicates that the quotient $|(P/Q)(\xi)|$ is bounded as $|\xi| \rightarrow \infty$ ξ real and $P \prec\prec Q$ that the same quotient goes to zero as $|\xi| \rightarrow \infty$.

Lemma 2.3. *Assume P and Q constant coefficients differential operators such that $P \prec Q$, Q hypoelliptic and $(P\varphi, Q\varphi) = 0$ for all $\varphi \in N(Q)^\perp$. Then $P \prec\prec Q$.*

3. GEOMETRIC SYMBOL IDEALS

We have the following problem: For which class of operators, do we have that the iterated symbols define hypoelliptic operators?

We will use notions and results from [18]. An ideal of holomorphic functions, in the sense of [18], is an ordered set of (f, δ) , where f is holomorphic in $\delta \subset \mathbf{C}^\nu$, that is closed for addition and multiplication with a holomorphic function, on the intersection of the domains corresponding to the functions. Two ideals are said to be equivalent, if for every point in the domain, any function in one ideal for this point, is in the other for the point. A local pseudo-base for the ideal (I) , is a finite system \mathcal{K} of holomorphic functions $F_i \in (I)$ such that for every $f \in (I)$, $f \equiv 0 \pmod{\mathcal{K}}$ in a neighborhood of a given point. If an ideal has a pseudo-base, then any equivalent ideal has a pseudo-base. A geometric ideal, is an ideal, defined as holomorphic functions and domains (f, δ) , such that $f \equiv 0$ on $\Sigma \cap \delta$, where Σ is a given characteristic set (analytic set).

Condition 3.1 (HE). *A class of holomorphic functions \mathcal{K} , is said to have the property (HE), if it satisfies the criterion on complex zero's for hypoellipticity, that is if $f \in (I)$ and ζ is such that $f(\zeta) = 0$ and $\text{Im } \zeta$ is bounded, then $\text{Re } \zeta$ is bounded.*

(cf. ([11]))

Lemma 3.2. *Given (I) , an ideal of holomorphic functions with the property 3.1, we have a local pseudo-base.*

Proof: The condition 3.1 defines a geometric ideal, why the Lemma follows from a theorem by H. Cartan [18] Ch VIII, sec.3 \square .

We will in this section study only entire functions of finite type. The ideals can thus be studied in the topology $\text{Exp}_{\rho,A}(\mathbf{C}^\nu)$ and we assume that ρ is chosen so that we have a product topology. Any $C^\infty(\mathbf{R})$ -function, real-valued and of exponentially finite (real) type, is the real part of an analytic function, in fact such a function is a real-analytic function itself. The same result holds in several variables for $\text{Exp } A$ with product-topology, by iteration of the one-dimensional result. Thus, given a real-valued function in $C^\infty(\mathbf{R}^\nu)$ of (real) exponential type A , we have $f = \text{Re } F$ in $\text{Exp } A$.

Assume $E(x, y)$ a kernel in $C^\infty(\Omega \times \Omega)$, $\Omega \subset \mathbf{R}^\nu$, that can be represented, by the real part of a holomorphic function f and thus has a representation in C^∞ by a real analytic function. Assume F has complex zero's (in both variables separately), that include the set of common complex zero's, for an ideal (I_{HE}) , defined by the property 3.1. Assume \mathcal{K} a local pseudo-base of holomorphic functions F_1, \dots, F_p for (I_{HE}) . According to Rückert Nullstellensatz ([18])

$$F^\lambda \equiv 0 \pmod{\mathcal{K}}$$

for some real number λ . This means, that we can determine holomorphic functions A_1, \dots, A_p in $H(\Omega \times \Omega)$, such that $F^\lambda = \sum_{j=1}^r A_j F_j$, in $H(\Omega \times \Omega)$.

As the regularizing operators are compact, we note that for reduced and regularizing operators the square root of the operator is defined. More precisely, since the operator ideal of compact operators is idempotent, if we consider a reduced $F \in H \cap L^2$ and $F = [F^{1/2}, F^{1/2}]$, we have that $F^{1/2}$ corresponds to a compact operator and if we use that $H \cap L^2$ is nuclear, we see that $F^{1/2} \in H \cap L^2$.

Assume F_1, \dots, F_p a pseudobase of hypoelliptic polynomials, $V = \cap_j Z_{F_j}$, then locally $Z_{F_1} \cap \Omega_1 = Z_{F_2} \cap \Omega_1$, for a domain of holomorphy Ω_1 . Since the pseudo-base elements are assumed hypoelliptic, it is sufficient to consider the real part $P_j = \text{Re } F_j$ and we have $P_1 = HP_2$ for instance with H real analytic in Ω_1 . Using parametrices to these operators, adjusted to \mathcal{E}' , we see that $E_2 = HE_1$, where E_j are entire and of type 0 for $j = 1, 2$. This means that, H is entire and of type < 0 (study $I = HP_2E_1$) and is thus bounded and by Liouville's theorem, constant (if the argument is divided into two inclusions, we get $H' = 1/H''$ constant) We conclude that $P_1 \sim P_2$ and thus $F_1 \sim F_2$. The same argument can be applied for any pseudo-base elements. We can also use this argument to prove that it is sufficient to consider (I_{HE}) as generated by hypoelliptic and equivalent (in strength) operators and over constant coefficients.

A lacunary point for an ideal (I) , is a point such that for all functions $f \in (I)$, f is not in (I) for this particular point. We have seen (cf. [8]) that for partially

hypoelliptic, constant coefficients polynomials, the set of lacunary points relative the corresponding ideal (I_{HE}) , is a (locally) finitely generated set.

The symbol E , can now be represented as

$$E = \frac{1}{2} [b_{\Gamma+} f + b_{\Gamma-} \overline{f}]$$

Over the lineality, we have that $E = \text{Re } f$ and outside this set we have that the difference is in $\text{Exp}_{0, \|\cdot\|}$ and in C^∞ . Particularly, if the symbol allows real support, the limit is well defined and uncomplicated. The set of complex lineality, can be defined as before, but we will have to prove a proposition analogous to b_Γ being portable by $\Delta_{\mathbf{C}}$.

4. LINEALITY FOR AN IDEAL

Assume the ideal (I) defined by a homomorphism h , that is $(I) = (\ker h)$. Consider now the lineality $\Delta(h)$, corresponding to h with $h \sim_\infty \text{Re } h$, where \sim_∞ refers to equivalence in strength (cf. [8]). If h is injective, naturally $\Delta(h(f)) = \Delta(f)$ for all $f \in (I)$. Otherwise $\tau_\eta h(f) = h(f)$ is equivalent with ${}^t\tau_\eta f - f \in J_h$ and we define $\Delta_I(h) = \{\eta \mid \tau_\eta f - f \in J_h \text{ for all } f \in (I)\}$. We assume in this context, that η defines a full line. Assume further that h maps real-analytical functions on real-analytical functions. Using the theory on analytic functionals (cf. [15]), if $b_\Gamma h$ is quasi-portable by $\Delta_I(h)$, then it is portable by $\Delta_I(h)$. Quasi-portability means particularly that the limit $\lim_{\Delta_I(h) \ni \eta \rightarrow 0} \tau_\eta h(f)$ exists, independently of the choice of $\eta \in \Delta_I(h)$ and assuming $\Delta_I(h)$ is analytic, we can conclude that $b_\Gamma h$ is portable by $\Delta_I(h)$. Strictly speaking, if the ideal is over $H(U)$, for an open set U , the porteur is $U \times \Delta_I$, but we will write the porteur as Δ_I , since b_Γ does not depend on U .

For a reduced F , algebraic over τ_η , the equality $\tau_\eta(BF) = \tau_\eta(B)\tau_\eta(F) = BF$ should be treated as $\tau_\eta B \tau_\eta = B$. Assume $B = I_{B'}$, that is given by an integral operator with kernel B' , then $\tau_\eta B = B\tau_{-\eta}$. We assume also, $B'(x + \eta, y - \eta) = B'(x, y)$, that is η is in the lineality for the kernel, separately in each variable $\eta \in \Delta_{x,y}(B')$. Particularly, if $\eta \in \Delta(h)$ and if $\ker \sigma_h$ gives the coefficients to $h(f) = 0$ on the form of integral operators B with kernel B' . Then, for all $B \in \ker \sigma_h$, we have $B'(x + \eta, y - \eta) = B'(x, y)$.

Assume f is not reduced, but partially hypoelliptic and $h(f)$ with the same properties. $h(f) = \sum_j A_j F_j$ with $A_j \in H$ and F_j reduced and $f = \sum_j B_j F_j$ with $B_j \in H$. Finally, $h(F_j) = \alpha_j F_j$ for constants α_j and all j . Then, $\eta \in \Delta(h(f)) \Leftarrow \eta \in \cap_j \Delta(A_j)$. If further $h(B_j) = \beta_j B_j$ for constants β_j and all j , then $\Delta(h(f)) = \Delta(f)$. Otherwise, $\frac{h(B_j)}{B_j} = \beta_j \in H$ not all constant, but with kernel β'_j . Thus, $\eta \in \Delta(h(B_j))$ implies that β'_j are invariant in both variables separately for translation with η . Finally,

$$\eta \in \Delta_{x,y}\left(\frac{h(B_j)}{B_j}\right) \Leftrightarrow \tau_\eta B_j - B_j \in J_h \quad \forall j$$

5. SCHWARTZ-TYPE TOPOLOGY

We make the following proposition, assume (I) a symmetric ideal of holomorphy with topology of Schwartz-type (cf. [15]) and a compact translation.

Proposition 5.1. *If $\psi \in (I)$ and $\psi \sim_m 0$ in $|\zeta|$ -infinity. Then we have that $\{d_\zeta \psi = \psi = 0\}$ is nowhere dense in $N(I)$.*

Proof:

Assume ψ is approximated by reduced $\psi_j \in (I)$ such that $d_\zeta \psi_j \neq 0$. Let $X = N(I)$ and assume ψ defined on an open, small, complex set $\Omega \supset X$. Assume further that $V_j = \{\zeta \in \Omega \mid \psi_j(\zeta) = \psi(\zeta)\}$ and that as $\psi_j \rightarrow \psi$, we have $V_j \uparrow \Omega$. In case V_j is connected, we have an infinite zero on V_j for $\psi_j - \psi$. Through the condition $\psi \sim_m 0$, if the $|\zeta|$ -infinity lies in a connected component in $\lim V_j$, then $d_\zeta \psi_j \sim_m d_\zeta \psi$, why the zero for ψ is simple in the $|\zeta|$ -infinity. \square

Assume $\psi(x) = \tau_x \phi / \phi$, where ϕ is considered on the real space. Let $\Gamma = \{x \mid \psi(x) = 1\}$ and let Γ_j be the corresponding sets for the approximating sequence ψ_j . As ψ_j are reduced, the sets Γ_j must be isolated points and as $\psi_j \rightarrow \psi$, we must have $\Gamma_j \rightarrow \Gamma$, why if $\lim V_j$ connected, the set Γ must be a point.

Note that a condition $\psi \rightarrow 1$ in the $|\zeta|$ -infinity should be compared with the condition of slow oscillation, when $\operatorname{Im} \psi \rightarrow 0$, why we always assume (I) symmetric. Note that (I) with compact translation can be seen as radical and thus as finitely generated, why we expect the minimally defined situation.

Lemma 5.2. *Assume existence of $\{\psi_j\}$ such that $\psi_j \rightarrow \psi$ and ψ_j reduced in (I) and such that $d_\zeta \psi_j \neq 0 \forall j$. Let $\Gamma = \{x \mid \psi = 1\}$. Then we have that Γ is an isolated point.*

Lemma 5.3. *There is an approximating sequence $\{\psi_j\}$ according to the previous Lemma.*

The second Lemma is immediate, since $(I_{RED}) \subset (I)$ and if the condition $\psi \sim_m 0$ is satisfied for ψ such that $\operatorname{Re} \psi \rightarrow 1$ and $\operatorname{Im} \psi \rightarrow 0$ as $|\zeta| \rightarrow \infty$, that is we are in the proposition considering the set $\{d(\psi - 1) = (\psi - 1) = 0\}$, then the approximating sequence exists as the translation is assumed compact.

Considering the first Lemma, if we do not have a connected component in the limit-point, there is at least one point in Ω where all the jets for $\lim \psi_j$ and ψ coincide. Assume now $d_\zeta \psi_j - \psi_1 \rightarrow 0$, with $d_\zeta \psi_j \neq 0$ and $\psi_1 \sim_m d_\zeta \psi$. We then have modulo monotropy that $\psi_1 \neq 0$ in the limit point, why ψ has a simple zero there. In the same manner we see that Γ is a point (modulo monotropy). Finally, we obviously have $\{\psi_1 = \psi = 0\}$ is nowhere dense in X , why the proposition follows modulo monotropy.

Consider the case with topology of Schwartz-type and weakly compact translation. More precisely, consider $\operatorname{rad} (I)$ and (I) is assumed as above with compact translation. If $(I) = \ker w$ for a homomorphism w , that we assume algebraic modulo monotropy, that is w is such that w^N is locally injective. According to proposition 5.1, if $\psi \sim_m 0$ and $w(\psi) = 0$, we have that

$\{d_\zeta w(\psi) = w(\psi) = 0\}$ is nowhere dense in $N(I)$. Further $\psi^N \in (I)$ means that $\psi \sim_m 0$ and $\{d_\zeta w(\psi^N) = w(\psi^N) = 0\}$ is nowhere dense in $N(I)$ and for a suitable N we see that $\{d_\zeta \psi = \psi = 0\}$ is nowhere dense in $N(I)$, where the argument is modulo monotropy.

6. LACUNARY POINTS

There will be significant lacunary cases in this approach. For instance any case, where b_Γ is not portable by the lineality, is lacunary for the symbolclass. More precisely, a lacunary case, is when the symbols $g_{(\alpha)}(it\eta)$, in the development of $g(\zeta) - g(x)$, are only locally constant in a neighborhood of 0. We study neighborhoods of Γ_g , such that there exists constants $C_{t,\epsilon}$

$$\Lambda_{t,\epsilon} = \{\eta \mid e^{-\epsilon\|\eta\|} \mid g(x + it\eta) - g(x) \mid \leq C_{t,\epsilon}\}$$

where $g(\zeta) - g(x)$ has real type 0. Obviously, for any finite t , these sets consist of all η . We can find a-priori-estimates in [15],

$\log \mid g_{(\alpha)}(it\eta) - g_{(\alpha)}(0) \mid \leq \gamma_\alpha(it\eta)$ where

$$\begin{aligned} \gamma_\alpha(it\eta) = & \frac{1}{(\pi\rho^2)^{n-k}} \int_0^\rho \int_0^{2\pi} \dots \int_0^{2\pi} \log \mid g_{(\alpha)}(it\eta + re^{i\theta}) - g_{(\alpha)}(re^{i\theta}) \mid \times \\ & \times d\theta_{k+1} \dots \theta_\nu dr_{k+1} \dots dr_\nu \end{aligned}$$

Immediately, $\gamma_\alpha(0) = 0$ and $\mid \gamma_\alpha(it\eta) \mid \leq \epsilon t$, for some small positive ϵ . Further, $\mid \gamma_\alpha(it\eta) \mid \leq \frac{t}{\rho} \log M_\alpha(t + \rho)$, where $M_\alpha(R) = \sup_{\|\eta\|_\infty \leq R} \mid g_{(\alpha)}(it\eta) - g_{(\alpha)}(0) \mid$ and as usual $M_\alpha(R)^{\frac{1}{|\alpha|}} \rightarrow 0$ as $\mid \alpha \mid \rightarrow \infty$, for all R . Finally, we have the following result (cf. [12] Cor. 4.4.14). The Lelong number $v_\varphi(\zeta)$ is defined for a pluri-subharmonic function φ on an open set $\Omega \subset \mathbf{C}^\nu$ as

$$v_\varphi(\zeta) = \lim_{r \rightarrow 0} \int \frac{d\mu(\zeta + w)}{c_{2n-2} r^{2n-2}}, \text{ where } d\mu = \Delta\varphi/2\pi.$$

Let $V = i\Gamma_g \cap \Omega$ and $\varphi = \frac{1}{2} \log \mid g(\zeta) - g(x) \mid^2$, then $v_\varphi(\zeta) = 0$ for $\zeta \in \Omega \setminus V$ and $v_\varphi(\zeta)$ on V gives the order of zero in ζ .

Note that in the case where $g_{(\alpha)}$ is only locally constant, we consider support-sets for $\widehat{g}_{(\alpha)}$ for $x \in \mathbf{R}^m$ on the form $x \in C_{[t_0, t_1]}^*$ if and only if $tx \in C_{[t_0, t_1]}^*$ for all t such that $t_0 \leq t \leq t_1$. Then obviously, $\widehat{g}_{(\alpha)} \in L^1(C_{[t_0, t_1]}^*)$. Further, for $\eta \in \Delta_{t,\epsilon}$

$$\mid g_{(\alpha)}(-it\eta) \mid \leq \left| \int_{C_{[t_0, t_1]}^*} e^{-i\langle x, it\eta \rangle} \widehat{g}_{(\alpha)}(x) dx \right| \leq e^{H_{t_0, t_1}(\eta)} \parallel \widehat{g}_{(\alpha)} \parallel_{L^1}$$

and using the norm for locally summable Fourier-transforms, M_β

$$(1) \quad \mid D^\beta g_{(\alpha)}(-it\eta) \mid \leq e^{H_{t_0, t_1}(\eta)} M_\beta(g_{(\alpha)})$$

where the last expression is well-defined, assuming that $g_{(\alpha)}$ is not constant on the support-set that we are considering. Particularly, using Cauchy's inequalities

$$\mid g_{(\alpha)}(it\eta) \mid \leq Mt^{-|\alpha|}$$

why $\mid g_{(\alpha)}(it\eta) \mid \rightarrow 0$ as $\mid \alpha \mid \rightarrow \infty$ and as $t \rightarrow \infty$. As $t_1 \rightarrow \infty$ in (1), we see that $g_{(\alpha)} \rightarrow 0$, so we can consider support-sets that are actually regular cones.

Conversely, we have to consider α larger than the Lelong-number on the support-cones, to avoid degeneracy in (1). We get expressions like

$$| D^\beta(g(\bar{\zeta}) - g(x)) | \leq C \sum_d^\infty e^{H_{t_0}(\eta)} M_\beta(F_{(\alpha)}) | x^\alpha |$$

for $F_{(\alpha)} = (g_{(\alpha)}(it\eta) - g_{(\alpha)}(0))$.

We say that x' preserves a constant value, if $F(x', y)$ is analytic for x' inner to a closed contour (assume product of contours) and y is outer to another closed contour, and if $F(x', y) = f(x', y) - f(x', x_n) \rightarrow 0$ as $y \rightarrow \infty$. Assume for instance C, C', C_1, C'_1 simple, closed contours in the respective planes x', x_n such that C' inner to C'_1 and C outer to C_1 . Assume F analytic for x' inner to C'_1 and y outer to C_1 , a a point outer to C'_1 and b a point inner to C_1 . Then according to Cousin ([6] §24, Prop. 4), for any small positive ϵ , there is a polynomial in $(x' - a)^{-1}, (y - b)^{-1}$, such that for x' inner to C , y' outer to C' ,

$$| F(x', y) - Q(\frac{1}{x' - a}, \frac{1}{y - b}) | < \epsilon$$

and we can produce estimates, for $\eta \in \Delta_{t, \epsilon}$ as

$$| f(x', x_n + it\eta) - f(x', x_n) | < \epsilon + \sum_\alpha A_\alpha t^{-\alpha}, \text{ for constants } A_\alpha \text{ and } x' \text{ inner to } C.$$

6.1. Lacunary points for the induced topology. Inducing a topology on the geometric ideal of operator symbols will affect the lacunary points for this ideal. In an attempt to understand this problem, we will modify the concept of a quasi-portable functional limit. T is said to be quasi-portable (with respect to X) by an open set $\Gamma \subset \Omega$, if there exists a $\mu_T \in H'_\Gamma(\Omega)$ such that $T = i_{\Gamma, \Omega}(\mu_T)$ where $i_{\Gamma, \Omega} : (X)'(\Gamma) \rightarrow (X)'(\Omega)$. Let $i^* : (X)(\Omega) \rightarrow (X)(\Gamma)$. If a measure μ with respect to X , is portable with respect to X by Γ , then

$$| \langle \mu_x, \phi \rangle | \leq C \| \phi \|_{X(\Gamma)}, \text{ for all } \phi \in (X)(\Omega),$$

why in this case the measure is also strictly portable (with respect to X) by Γ .

Assume more generally $\Gamma_{\mathbf{R}}$ a real, convex set. According to the theory of analytic functionals, if Δ_Γ is the indicator to $\Gamma_{\mathbf{C}}$ and $h_F(\eta) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log | F(t\eta) |$, for η real close to Γ , $|\eta| = 1$ and $F = \mathcal{F} T$ (the Fourier-Borel transform) to $T \in H'$ and for $\Gamma_{\mathbf{C}} = \Gamma_{\mathbf{R}} + i\Gamma_{\mathbf{R}}$ which we assume closed, $(i\Gamma_{\mathbf{R}})^\circ = (i\Gamma_{\mathbf{R}})^\perp$.

Assume μ a measure with respect to $X = H \cap L^2$, that is

$$(2) \quad | \mu(\phi)(x) | \leq C \| \phi \|, \quad \forall \phi \in X(\Omega)$$

Ω an open set in \mathbf{R}^ν containing x . Let $\Gamma_y = \{ty, t \in \mathbf{R}\}$ and

$$\Gamma_C = \{y, | \mu(\phi)(x + i\Gamma_y) | \leq C \| \phi \| \quad \forall \phi \in X(\Omega + i\Gamma_y)\}.$$

We can assume $\Omega = \mathbf{R}^\nu$ and $T^{\Gamma_y} = \Omega + i\Gamma_y$. Over Γ_C we obviously have

$$(3) \quad \lim_{t \rightarrow 0} \mu(\phi)(x + ity) = \mu(\lim_{t \rightarrow 0} \phi)(x + ity)$$

and we claim that $b_\Gamma \mu$ can be considered as quasi-portable by Γ , the union over all finite constants C in (2). The limits (3) are trivial over the lineality-set, that is $\Delta_\phi = \{y; \mu(\phi)(x + ity) = \mu(\phi)(x) \forall t \forall x\}$. Outside Γ_C in a conical neighborhood, $\Upsilon \subset \Gamma$, since the limit is trivial in $X(T^\Upsilon)$, we have

$$| \lim_{t \rightarrow 0} \mu(\phi)(x + ity') | \leq C' \| \phi \| \quad \forall \phi \in X(T^\Upsilon),$$

why the limit on the left side of (3) exists, for a new constant C' . Assume finally, that μ is reduced in L^2 , then for all $\phi \in X(T^\Upsilon)$ and $(\tau_{y'}\mu)(x) = \mu(x + ity')$ for an $y' \in \Upsilon \setminus \Gamma_C$, $|(\tau_{y'}\mu)(\phi)(x)| \leq C'' \|\phi\|$ and $\phi \in X(T^\Upsilon)$, where we have used that the translation will be bounded in X . We see that to any $\Upsilon \subset \Gamma$, there is a translated measure, defining the same functional limit $b_\Gamma \mu$.

Proposition 6.1. *Assume μ a measure with respect to $L^2 \cap H$, then $b_\Gamma \mu$ is quasi-portable by the lineality for μ . If μ is a measure as above, such that also $\hat{\mu}$ is a measure with respect to $L^2 \cap H$, we have for a $\phi \in L^2 \cap H$ and $\mathcal{F}T = \hat{\mu}(\phi)(x)$, T is semiportable by $\Gamma_C = \Gamma_{\mathbf{R}} + i\Gamma_{\mathbf{R}}$, the lineality to $\hat{\mu}(\phi)$ when $\|\phi\| \leq 1$.*

If for a closed, convex set Γ , the functional T is portable by every convex neighborhood of Γ , then T is said to be semi-portable by Γ .

Lemma 6.2. *T is semi-portable by Γ_C if and only if $h_F(x + ity) = 0$ for $iy \in (i\Gamma_{\mathbf{R}})^\circ$ or if and only if $h_F(x + ty) = 0$ for $y \in (\Gamma_{\mathbf{R}})^\circ = (\Gamma_{\mathbf{R}})^\perp$.*

It is thus, for our application $X'^{(0)}$ of no significance if we consider real or complex translations.

Assume now $F = \lim_{t \rightarrow 0} [F(x + ty) - F(x)]$, where F is for instance $\hat{\mu}(\phi)(x)$, for a measure μ , such that $\hat{\mu} \in X'^{(0)}$. Assume Γ the lineality to $\hat{\mu}(\phi)$, if $\|\phi\| \leq 1$, we have $h_F(x + ty) = 0$, for $y \in \Gamma$, for all x , why T is semi-portable (with respect to X) by Γ_C . If $h_F(\eta) \leq 1$ for all $\eta \in \delta\Gamma$, where $\delta\Gamma = \{y^* \mid \text{Re } y^* < y^*, y^* = 1 \mid y \in \Gamma\}$, then T is portable by Γ and $h_F(\eta) \leq \Delta_\Gamma$ for all real η . Finally, if i^* is defined as before with $\Omega = \mathbf{R}^\nu$ and $\mu \in \mathcal{C}'(\mathbf{C}^\nu)$ is such that $\hat{\mu} \in X$ and $\mu = \mu_1 + \mu_2$ with $\|i^*\hat{\mu}\|_{X(\mathbf{R}^\nu)} = 0$. We then have that $\|\hat{\mu}\|_{X(\mathbf{R}^\nu)} = \text{constant}(=1)$ implies existence of a $\hat{T} \in (X)'(\Gamma_C)$, such that $i^*_C(\hat{\mu}) = \hat{T}$. Further,

$$|\langle \hat{\mu}, \phi \rangle| \leq C \|\hat{\mu}\|_{X(\mathbf{R}^\nu)} \|\phi\|_{X(\mathbf{R}^\nu)} \leq C' \|\phi\|_{X(\mathbf{R}^\nu)}.$$

Thus, $\hat{\mu}$ is a measure with respect to X , strictly portable by Γ_C .

7. A GLOBAL BASE FOR PARTIALLY HYPOELLIPTIC OPERATORS

Define for $F_j \in \mathcal{K}$ $j = 1, \dots, p$, a set

$$G_\Omega = \{(x, F_j(x)) \mid d_\Gamma f \leq 1\}$$

where, $d_\Gamma f = b_\Gamma f - f$, can be defined on real or complex arguments. Define an ideal (I_{const}) over G_Ω , as $g = b_\Gamma f - \sum_j A_j F_j$, that is $b_\Gamma f \equiv g \pmod{\mathcal{K}}$ (assume for now that we are dealing with entire functions) and where $const$ is referring to constant coefficients A_j in the pseudo-base development. The zero's for (I_{const}) are given as points such that $b_\Gamma f \equiv 0 \pmod{\mathcal{K}}$. We then have that G_Ω can be approximated from the outside, by polynomially defined sets. Further, $f \equiv d_\Gamma f \pmod{\mathcal{K}}$ where according to Lemma 20.1, $d_\Gamma f$ is of type 0. Over the set of lineality, we have $f \equiv g \pmod{\mathcal{K}}$. Further, it is possible to construct a local pseudo-base for the projection of (I_{const}) on the first variable in G_Ω . If f is of type A and g is developed in a local pseudo-base for (I_{const}) , according to the

remainder theorem, it can be given as a polynomial in F_j 's with holomorphic coefficients of type A .

Consider, for this situation, lacunary points as points such that $d_\Gamma f \notin (I_{HE})$. Note that if $f \in (I_{HE})$ is of type $A > 0$, then f is reduced. That is if $f^{(\alpha)}(it\eta)$ is constant $\neq 0$, for all α , then $\eta = 0$. Since assuming $\eta \neq 0$, means $\sum_{\beta \neq 0} c_\beta (f^{(\alpha\beta)}(0))t^\beta \neq 0$, for some β , thus $f^{(\alpha)}$ is not a constant $\neq 0$. If on the other hand $f^{(\alpha\beta)}(0) = 0$, for all β , the type is 0!

The Cousin integral ([18]) is defined as

$$\Phi(x, y) = \frac{1}{2\pi i} \int_L \frac{f(x, z)}{z - y} dz$$

where x is assumed as inner to $\gamma = \gamma_1 \times \dots \times \gamma_n$, where γ_j are closed contours in respective plane, where L is a line containing the point y_0 in the y -plane (one-dimensional). L is assumed inner for a closed contour Γ and f is assumed analytic (regular) for (x, y) inner to (γ, Γ) . Assume l a line through y_0 , which does not intersect L in any other point and which is inner for Γ . Then Φ is analytic (regular) for x inner to γ and y not on L , but otherwise arbitrary. The analytic continuation $\tilde{\Phi}$ can be shown to be analytic in y_0 and on l , on one side of L . $\tilde{\Phi} = \Phi \pm f$ (cf. [6]).

Proposition 7.1. *Assume $\Delta_{\mathbf{C}}$ the lineality set and $r(I_{HE})^A$ a radical ideal of holomorphy taken over entire functions of type $A > 0$. Then there exists an ideal of holomorphy J_A , with the zero's given by $\Delta_{\mathbf{C}}$, such that $J_A \sim r(I_{HE})^A$.*

Proof: Assume $g \in J_A$, since this is a geometric ideal, we have $g = \sum_j c_j \phi_j$ and we assume that in a sufficiently small neighborhood of $\Delta_{\mathbf{C}}$, $\phi_j \in r(I_{HE})$, for all j . Thus $g^t \in (I_{HE})$, for large t . Conversely, if $g \in r(I_{HE})$ and $g = 0$ on $\Delta_{\mathbf{C}}$, then $g^t = 0$ on $\Delta_{\mathbf{C}}$ and $g^t = \sum_i a_i F_i$ in a neighborhood of $\Delta_{\mathbf{C}}$. We may assume $F_i = 0$ on $\Delta_{\mathbf{C}}$, for all i , which gives a development of g^t in the pseudobase for J_A , which is a radical ideal. \square

Assume g entire and in J_A and that $g \rightarrow 0$ on a line L_1 . Assume L_2 another line in the same coordinate-plane, it is then a corollary to Lindelöf's theorem, that $g \rightarrow 0$ on the sector between and including L_1 and L_2 . That is, since g is of type 0, there is a F analytic for $1/|z| > \epsilon$ such that in this region, $zg(z) = F(\frac{1}{z})$ and the corollary can be applied on $zg(z)$. The corresponding Cousin-integrals

$$\varphi_j(x, y) = \frac{1}{2\pi i} \int_{L_j} \frac{g(x, z)}{z - y} dz \quad j = 1, 2$$

have the property that $\varphi_1 + \varphi_2 = 0$. The argument can be repeated, so that $\Phi = \sum_p \varphi_p = 0$ describes the entire $\Delta_{\mathbf{C}}$. According to Cousin (cf. [6]) there is a "global" function which coincides with φ_p on the corresponding inner sector for each p and such that $\tilde{\Phi} - \Phi$ is analytic. We claim that the second Cousin-problem is solvable in this situation and that the ideal J_A has a global pseudobase. Particularly, if $\Delta_{\mathbf{C}} = \{0\}$, we have a global pseudo-base for (I_{HE}) .

8. THE RÜCKERT NULLSTELLENSATZ

We have established that $\operatorname{Re} P \in \mathcal{H}_+$, where \mathcal{H}_+ denotes \mathcal{H}_c for positive c and $\operatorname{Im} P \prec\prec P$ implies $P \in \mathcal{H}_+$, but the converse does not hold, that is $P \in \mathcal{H}_+$ does not imply that $\operatorname{Im} P \prec\prec P$, more precisely $I_+^* \subset I_+ \subset r(I_+)$. According to the real version of Nullstellensatz, the given ideal I is radical if and only if $P_1^2 + \dots + P_k^2 \in I$ implies $P_i \in I$, for all i . This is consistent with our results on hypoellipticity for self-adjoint polynomial-operators in $L^2(\mathbf{R}^\nu)$. Particularly, for a radical ideal of real symbols, locally $|f|^2 \in I \Rightarrow \operatorname{Re} f, \operatorname{Im} f \in I$ and in our case, the ideal is locally given by real zero's. If for an ideal I , we let $I^* = \{ \operatorname{Re} f \mid f \in I \}$ and $I_{\mathbf{C}} = I^* + iI^*$ the "complexified" ideal, it not difficult to see that $I_{\mathbf{C}} = I$ if and only if I is radical. For symbols as entire functions, we give the following result

Proposition 8.1. *Assume $F = (F_1, \dots, F_p)$ a global pseudo-base for $I_{HE}^{*,A}$, the ideal of hypoelliptic symbols, self-adjoint and of type A and $V = \cap_j N(F_j)$. If f is an entire function of type A , such that $f = 0$ on V , then according to Rückert's Nullstellensatz, we can find constants A_1, \dots, A_p , such that*

$$(\operatorname{Im} f)^\lambda = \sum_j A_j F_j \quad \text{in } \Omega$$

for Ω a holomorphically convex neighborhood of V and for $\lambda > 0$ real.

In the polynomial case, assuming $\operatorname{Re} f \in I_{HE}^{*,A}$ and $\lambda > 1$, for f as in the proposition, we have $f \in I_{HE}^A$. If $I_{PHE}^{*,A} = r(I_{HE}^{*,A})$ then this must be a radical ideal, and it is locally defined by its real zero's.

9. THE WHITNEY CLOSURE

Assume, in analogy with Kohn's approach ([13]), $\alpha(T) = \frac{\psi(\zeta+T)}{\psi(\zeta)}$. For an analytic homomorphism b , $b(\psi)(\zeta + T) = \frac{1}{\alpha(T)} b(\psi)(\zeta) + C_T$. If $C_T = 0$ we know that α is holomorphic and non-constant on a bounded set, symmetric around the origin. Let $\Sigma(b(\psi)) = \{T \mid b(\psi)(\zeta + T) = b(\psi)(\zeta) + C_T\}$. Define the Whitney-closure, $\overline{(I_{C_T=0})} = \{\psi \mid b(\psi)(\zeta + T) = b(\psi)(\zeta) + C_T\}$.

Proposition 9.1. *There is an ideal (J) such that $\operatorname{rad}(J) \sim \overline{(I_{C_T=0})}$*

Let $\Omega_0 = \{T \mid b(\psi)(\zeta + T) = b(\psi)(\zeta)\}$ with $\Omega_0^{(j)} = \{T \mid b(\psi^j)(\zeta + T) = b(\psi^j)(\zeta)\}$. We can prove that $\Omega_0^{(j)} \downarrow \{0\}$ as $j \uparrow \infty$. Consider $\psi = 0$ on $\Omega_0^{(j)}$, then $\psi \in I(\Omega_0^{(j)})$, where $\Omega_0^{(j)} = N(J)$ for some J and $\psi \in \operatorname{rad}(J)$. Add a constant, such that $\psi - c = 0$ on $\Omega_0^{(j)}$, this gives the result (modulo monotropy). We will now prove a correspondent to regular approximations of periodic points.

Proposition 9.2. *Consider the sets $V = \{T \mid B(\zeta + T) = B(\zeta)\}$ and $V' = \{T \mid DB(\zeta + T) = DB(\zeta)\}$, where $D = D_{\zeta_1} \dots D_{\zeta_n}$. In the finitely generated case, we have that $V \cap V'$ nowhere dense in V' . This means that there are $T_j \in \Sigma(B)$ such that $B(\zeta + T_j) = B(\zeta) + c_j$ and $0 \neq c_j \rightarrow 0$ as $T_j \rightarrow T_0$ and $T_0 \in V$.*

Proof:

Assume (J) with Schwartz-type topology and with a weakly compact translation, in the sense that we assume τ_T is compact over (J) and weakly compact over

$rad(J)$. Let $B = h(f)$, $f \in (J)$ and reduced, then modulo monotropy, $B^2 \sim_m h^2(f)$ and if $h^N = 1$ for $N \geq N_0$ we see that τ_T is compact over an iterate of B . Further $\tau_T B = h(\tau'_T f)$, where $\tau'_T = \tau_{T'}$ is also a translation. For $T \in V'$ we have that $B(\zeta + T) - B(\zeta) = c_T$ for all $\zeta \in \mathcal{B}$, a bounded set in Ω pseudoconvex and where the constant c_T is dependent on T . Assume further $T \rightarrow T_0 \in V$. Our proposition is that there is a sequence of T_j in V' such that $T_j \rightarrow T_0$. For h algebraic, the result follows from the conditions on f and proposition 9.1. For $h = \tau_\epsilon w$ and w algebraic, there is a T_j , $\tau_{T_j} B^2 - B^2 = C_{T_j} \rightarrow 0$ as $T_j \rightarrow T_0$. Modulo monotropy, this means that $h(\tau_{T_j} B - B) \sim_m C_{T_j} \rightarrow 0$. The condition $T_j \in V'$ means that $\tau_{T_j} B - B = C'_{T_j}$ and we have $h(C'_{T_j}) \sim_m C_{T_j} \rightarrow 0$ as above. \square

10. A MONOTROPY-IDEAL

Cousin [6] uses the concept of monotropy as a weak sufficiency condition for solving the Cousin-problems. We will develop some results for pseudo-differential operators using this concept. Assume A a set of isolated points, for a continuous function g , then there is a holomorphic function, with zero's on $A \cap U$, for an open set U . Let A_ϵ be a set such that $P \in A_\epsilon \cap U \Rightarrow \exists P' \in A \cap U$, such that $|P - P'| < \epsilon$. Let f_ϵ be the holomorphic function that has zero's on $A_\epsilon \cap U$, we use the notation $f_\epsilon \sim_m g$.

Rouché's theorem can be used to define a "monotropy" ideal J_h^ϵ such that, $f \in J_h^\epsilon$, if there exists a homomorphism g , such that

$$|h(f) - g(f)| < \epsilon |h(f)| \text{ on } \delta\Omega$$

for a small $\epsilon > 0$, where Ω is a small disc-neighborhood of a point in $N(J_h)$. This condition means that $h(f)$ and $g(f)$ have the same number of zero's (counted with multiplicity) in Ω . The condition particularly means that $h(f) \sim_m g(f)$. Assume further that g is an algebraic homomorphism, $g^2(f) = g(f^2)$ that commutes with h , such that $g(f) - f^2 \in J_h$, that is $h(f^2) = hg(f)$. If we have $h(f) - g(f) \in J_h$, then $h(f^2) = h^2(f)$. Otherwise, if $f \in J_h^\epsilon$ with $Z_f \subset Z_{h(f)}$, $h^2(f) - h(f^2) = h(h(f) - g(f)) \sim_m 0$, that is $h^2(f) \sim_m h(f^2)$ over J_h^ϵ . In the simplest case g can be chosen as the identity operator.

Given our analytic homomorphism h over an ideal (I) , there is an algebraic homomorphism g , such that locally $h \sim_m g$ over (I) . Assume h is a homomorphism with $Z_f \subset Z_{h(f)}$ and h^N injective. Further, $\Omega = \text{nbhd } Z_f$ and $h(f) \neq 0$ on $\delta\Omega$, where $\Omega = \Omega(R)$ and $1/R < 1 + \sum_j |\alpha_j|$, where $h^j(f) = \alpha_j f$. Assume further that R is so large that $RC > 1/\epsilon$, for a given $\epsilon > 0$, where C is such that $|h^N(f) - f| = C |f|$. Thus, there exists an open set $\Omega = \Omega(R)$ such that $|h(f) - f| < \epsilon |f|$ on $\delta\Omega$. If $\alpha = h(f)/f$ is holomorphic on $\delta\Omega$ and $|\alpha| > \epsilon$ on $\delta\Omega$, the existence of an algebraic homomorphism (dependent on Ω) follows.

For h such that h^2 is injective, we have existence of a square g with respect to h , $h^{-1}gh = \alpha$, for $\alpha(f) = f^2$. Thus, $h^{-1}g^2h = \alpha^2$ and so on. If $h\alpha = \alpha h$, we have $\alpha^{-1}g = \sqrt{g} = 1$. Again $h(f) - g(f) = h(f) - f^2$ and $h(f) - g(f) \in J_h \Leftrightarrow h^2(f) = h(f^2)$. Let

$$I_f^\epsilon = \{g \mid \tau_\epsilon g = 0 \text{ on } Z_f \cap \Omega \text{ } \exists \epsilon \text{ small}\},$$

where τ_ϵ denotes translation and Ω is a small domain of holomorphy around Z_f . If h has integral representation, we have $h: I_f^\epsilon \rightarrow I_f^{-\epsilon}$ continuously. This means that h does not spread zero's much. For h such that $h(f) \sim_\infty \text{Re } h(f)$ with h^2 injective and f reduced, we have $h(f)^2(\zeta) = 0$ implies $h(f)(\zeta) = 0$. Assume g the square with respect to h as above and $h^2(f) = g(\tau_\epsilon f)^2$, then $h^2(f) = 0$ implies $f = 0$. In the same way $\tau_\epsilon h^2(f) = 0$ implies $\tau_\epsilon f = 0$, why $\epsilon = 0$. Note that h^2 is reduced if and only if g^2 is reduced. Thus, with these conditions, $h^2(f) = h(f)^2$.

Proposition 10.1. *Assume h a homomorphism with $\text{Re } h \sim_\infty h$ and h^N injective, for $N > N_0$, N_0 some positive number. Then we have over J_h^ϵ , that h is an algebraic homomorphism.*

Proof:

The condition $f \in J_h^\epsilon$ means that there exists an algebraic homomorphism g , such that $h^{N-1}(f) = g^{N-1}(\tau_\epsilon f)$. Note that ϵ depends on both f and ζ . Then $h^N(f) = g^{N-2}(\tau_\epsilon^2 f^2) = g^N(\tau_\epsilon f)$ means that $\epsilon = 0$, thus $h^N(f) = h(f)^N$. \square

Define $(I_h^m) = \{f \mid h(f) = \tau_\epsilon g(f)\}$ for an algebraic homomorphism g and for some ϵ . We are assuming $h\tau_\epsilon = \tau'_\epsilon h$, where also τ'_ϵ is assumed to be translation. Let $B_\epsilon = \{f \mid \tau_\epsilon f/f \text{ not constant}\}$. We form (I_h^m) over $(I) = \{f \mid B_\epsilon \text{ bounded and symmetric with respect to the origin}\}$. We are assuming the translation (weakly) compact over (I) , why $g(B_\epsilon)$ has the same properties over (I) . If $f \in (I_h^m)$, we have $\frac{\tau'_\epsilon f}{f} = \frac{g^{-1}h(f)}{f}$. Further, if $h(f) \sim_m g(f)$, we have (with a different ϵ) $h(f)^2 \sim_m g(f^2)$, that is the representation is invariant for iteration. Assume $h(f^2) \sim_m h(f)^2$ (algebraic translation), then $f^N \in (I_h^m)$ implies $f \in (I_h^m)$, that is we have a radical ideal.

11. MEASURE ZERO DOMAINS

We know that polynomial operators are never regularizing as pseudo-differential operators and this means that necessarily polynomials do not have an infinite zero in the infinity (it has not complex dimension). Assume our system has polynomial right hand sides and consider an analytic surface $S = \{y = G(x)\}$, for a polynomial G . If $\int_{(I)} (1 + |G'(x)|^2) dx = 0$ and if (I) is a neighborhood of the origin, we have that the measure for (I) is zero. If $\int_{(I)} |G'(x)|^2 dx = 0$, then as we shall see, there is a segment in (I) such that $G'(x) = 0$ on this segment. Then according to Hurwitz theorem, $G'(x) \equiv 0$ on (I) . The conclusion is thus $\int_{(I)} dx > 0 \implies \int_{(I)} |G'(x)|^2 dx > 0$. Thus for all polynomials we have that the zero-set does not have positive area.

Particularly, assume w is a polynomial in x, y given our conditions and $\int_{(\tilde{I})} w d\sigma = \int_{(\tilde{I})} Q_x - P_y d\sigma$ (we assume $\int_{(I)} w dx dy = \int_{(\tilde{I})} w d\sigma(t)$). Given a finite integral on both sides, we see that there exist $\omega_j \neq 0$ such that $\omega_j \rightarrow \omega$ uniformly on compact sets. Further, that if (I) is a neighborhood of the origin

$$\int_{(\tilde{I})} \omega d\sigma = 0 \implies \sigma(\tilde{I}) = 0$$

Note also that if $\eta(x) = h(x)/x$ and $\int_{(I)} (1 + |\eta'(x)|^2) dx < \infty$, we have that the surface $\{y = \eta(x)\}$ is normal. However, if $\int_{(I)} |x\eta'(x)|^2 dx = 0$ and $\int_{(I)} dx > 0$,

according to Hurwitz theorem, we have that $x\eta'(x) \equiv 0$ in (I) and $\frac{dh(x)}{dx} \equiv \frac{h(x)}{x}$ in (I) .

Assume Ω an open subset of a Stein-variety V . We can define the quasi-porteur corresponding to an analytic functional in $H'(V)$ as an open set where the functional has a local representation. A porteur is a set where we have representation of $T \in H'(X)$ by a measure with compact support. If $\Omega \subset X$ is a quasi-porteur for $T \in H'(X)$, we have for all neighborhoods W of Ω , existence of a measure μ_W with compact support in W , such that $T(\phi) = \int_W \phi d\mu_W$ for all $\phi \in H(X)$. If Ω is a complex-analytic quasi-porteur for σ , we can determine a measure μ with compact support on the boundary of Ω , Γ' , such that

$$(4) \quad \int_{\Omega} \phi d\sigma = \int_{\Gamma'} \phi d\mu \quad \phi \in H(X)$$

such that $|\int_{\Gamma'} \phi d\mu| \leq C \sup_{\Gamma'} |\phi(x)|$ for $\Gamma' \supset \text{supp } \mu$. If T is represented by μ in H' , we can say for the restriction to a line L , that $\hat{T}(\zeta) = 0$ means that $\mu|_L = \text{const.} \delta_L$. In order to, starting from a dynamical system determine the porteur, we note that every subset W' such that $\int_{W'} d\sigma = 0$, is a subset of the quasi-porteur. If we limit ourselves to the case with analytic Ω 's, this is also a subset of the porteur.

12. SYMMETRY

Consider $I_{\Gamma}(F) = \int_{\Gamma} F(x, y) dx$ and $I_{\Gamma}^*(F) = \int_{\Gamma} \overline{F}(x, y) dy$. Assume for a domain $\Omega \subset (I) \times (I)$ and $\Gamma = b\partial\Omega$

$$\int_{(\Omega)} \frac{d^2 F}{dy dx} - \frac{d^2 F}{dx dy} dx dy = \int_{\Gamma} \frac{dF}{dx} dx + \frac{dF}{dy} dy = 0$$

This means that $\frac{d^2 F}{dy dx} = \frac{d^2 F}{dx dy}$ over (Ω) . Further, $I_{\Gamma}(F_x) + I_{\Gamma}^*(\overline{F_x}) = 0$ under the condition above, why if \overline{F} is holomorphic, we have $I_{\Gamma}(F_x) = 0$ and F is constant over (Ω) . Note that if $I_{\Gamma}(F) = 0$ for a polynomial in x , F , then outside (Ω) , we have the polynomial case in y . Further, we can define the operators $T_{\Gamma}(u) = \int_{\Gamma} F(x, y) u(y) dy$ and $T_{\Gamma}^*(v) = \int_{\Gamma} \overline{F(x, y)} v(x) dx$. We can define a normal operator as an operator such that the image of a bounded sequence of x has a subsequence that converges uniformly on compact sets.

Lemma 12.1. *If, for a holomorphic function F , we have that $\int_{(\tilde{I})} F d\sigma = 0$ we have that one of the following propositions is true, either $\int_{(\tilde{I})} d\sigma = 0$ or $F \equiv 0$ on (\tilde{I}) .*

In the polynomial case, we have seen that if (Ω) is a neighborhood of the origin, we have that the first condition holds. Note that if $\int_{(\Omega)} \omega dx dy < \infty$ and assuming ω polynomial in x, y , then also the measure for singularities must be zero. If F is holomorphic and $\int_{(\tilde{I})} d\sigma = 0$, if also $\int_{(\tilde{I})} F d\sigma = 0$, we must have that F does not have an infinite zero in $(\Omega) \ni 0$ according to Hurwitz. Does F have to be a polynomial?. If $F \equiv 0$ on (Ω) , we must have $F \in \text{rad } I(\Omega)$, so F is a polynomial in x, y . More precisely, for $t = t_1 + it_2$, if t_1, t_2 are reduced, we have that

$\int_{(\tilde{I})} d\sigma(t) = 0$ implies that (\tilde{I}) is algebraic. According to Nishino (cf. [16]), these sets are removable, for analytic continuation.

If $(I) \ni f_j \rightarrow f \in (\tilde{I})$ with uniform convergence on compact sets, then $N(f_j) \rightarrow N(f)$ (cf. [16]). We will consider the sets $V_j = \{\zeta \mid \int_{(\Omega)} f_j dx dy = 0\}$ and $\tilde{V}_j = \{\zeta \mid \int_{(\tilde{I})} f_j d\sigma(t) = 0\}$. For f_j we thus have that either $\int_{(\tilde{I})} d\sigma(t) = 0$ or $f \equiv 0$ on (\tilde{I}) . If $dL_T \sim_m 0$, we know that there are T_j such that $\frac{dL_{T_j}}{dx} dx + \frac{dL_{T_j}}{dy} dy \rightarrow 0$ as $j \rightarrow \infty$ and given that $\frac{d^2 L_{T_j}}{dx dy}, \frac{d^2 L_{T_j}}{dy dx}$ continuous, $\int (\frac{d^2 L_{T_j}}{dx dy} - \frac{d^2 L_{T_j}}{dy dx}) dx dy \rightarrow 0$ as $j \rightarrow \infty$. Thus, if T_j is chosen such that the integrand holomorphic, we have that $(\tilde{\Omega})$ is a domain for symmetry. Assume $\int_{\tilde{I}} d\sigma(t) > 0$, then we have through the lemma that $\tilde{V}_j \subset N(f_j)$. Further, if the integral is taken over $N(f_j)$, we must have that $\int_{\tilde{I}} f_j d\sigma(t) = 0$. This corresponds to convergence $\tilde{V}_j \rightarrow \tilde{V}$ as $j \rightarrow \infty$.

We can form the ideals $(J) = \{f \mid \int_{(\tilde{I})} f d\sigma(t) = 0 \text{ on } \tilde{V}\}$ and the ideal under topology for normal convergence, where the measure that involves an algebraic homomorphism, is considered as compact. Existence of a normal and regular approximation, gives according to Hurwitz theorem the same conclusions and we denote the corresponding ideal (J^*) , which is considered as weakly compact. More precisely, assume $I_{(I)}(w_j) \rightarrow I_{(I)}(w)$, uniformly on compact sets in ζ . Further, if $w'_j \rightarrow w$ as a normal and regular approximation, then $I_{(I)}(w) = \lim_j I_{(I)}(w'_j) = \lim_j I_{(I)}(w_j)$. We must have that $\lim_j I_{(X)}(w - w'_j) \neq 0$ implies $\sigma(X) = 0$. Assuming w algebraic in x, y , we must have that $W = \{(x, y) \mid \lim_j w_j(x, y) = w(x, y)\}$ has measure zero. For instance, let $w_j(x, y) = w(x_j, y_j)$. In the same manner, $W' = \{\zeta \mid \lim_j w_j(x, y)(\zeta) = w(\zeta)\}$ has measure zero with respect to a relevant measure $\exists F \int_W d\sigma(t) = \int_{W'} F(d\zeta)$. Note that since the integral is invariant for monotropy, we do not necessarily have that the $I_{\Omega}(w)$ are holomorphic.

Assume F holomorphic and F_1 continuous and $F_1 \sim_m F$ on (I) and that $\int_{(\tilde{I})} (F_1 - F_j) d\sigma(t) \rightarrow 0$ as $j \rightarrow \infty$, for F_j a holomorphic regular (in the sense of [16]) approximation. Since the integral is invariant for monotropy, we have $\int_{(\tilde{I})} (F - F_j) d\sigma \rightarrow 0$ as $j \rightarrow \infty$, why if $F_1 \neq 0$ on (I) it follows that $\int_{(\tilde{I})} d\sigma(t) = 0$

Proposition 12.2. *Assume T_{Γ} defined as above with $T_{\Gamma}^* T_{\Gamma} = T_{\Gamma} T_{\Gamma}^*$ and $\int |F|^2 d\sigma < \infty$, then T_{Γ} is a normal operator.*

Now define the operators

$$T_{\Gamma}(u) = \int_{\Gamma} G(x, y) u(y) dy \text{ and } T_{\Gamma}^*(u) = - \int_{\Gamma} \overline{F}(x, y) u(x) dx$$

so that $T_{\Gamma}(1) - T_{\Gamma}^*(1) = \int_{\Gamma} G dy + \overline{F} dx$. If F, G are real and $\int_{\Gamma} F dx + G dy = 0$, then $-dG/dx = dF/dy$ and through mass-conservation $dG/dy = -dF/dx$. Thus, $dG/dT = dF/dT$. If $G_x \sim_m F_y$, then $\int_{\Gamma} F dx + G dy = 0$ and if $G = {}^t F$ with the notation $(F \frac{d}{dy})^t = \frac{d}{dx} {}^t F$, so that $G_x - F_y = (F \frac{d}{dy})^t - (\frac{d}{dy} F) \sim_m 0$, we have that

$\langle T_\Gamma(u), v \rangle = \langle u, T_\Gamma(v) \rangle$. Define $S_\Gamma^*(u, T) = - \int_\Gamma F(x + T, y)u(x)dx$ and $T_\Gamma(u) = \lim_{T \rightarrow 0} S_\Gamma$.

Assume symmetry in the tangent space $TS = \{\frac{d}{dx}G \sim_m \frac{d}{dx}M\}$, for M real, that is $G \sim_m M + iC(y)$, where C is independent on x . For instance, if $F - F^* \in (I)(\Omega)$ where Ω is a domain for symmetry, we can form the closure $(\bar{I})(\Omega)$, then we have symmetry in the tangent space over Ω . If $G = {}^tF$, we thus have $\int_\Omega (\frac{d^tF}{dx} - \frac{dF}{dy})u \otimes v d\sigma = 0$, for all $u, v \in H(X)$, for ${}^tF, F$ in TS . Thus, $S_\Gamma = S_\Gamma^*$ and T_Γ can be considered as the limit of self-adjoint operators.

Assume $0 < \int_\Omega d\sigma < \infty$ and $\int F d\sigma < \infty$. Assume $\int_\Omega (F - A)d\sigma = 0$ for a constant A . Further, $F(\phi) = \int_\Omega F(x, y)\phi d\sigma$ and $FF^* = F^*F$. We can prove that $F^*(\phi) = \bar{A} \int_\Omega \phi d\sigma$. If $\int_\Omega |F|^2 d\sigma < \infty$ then $S = \{F\bar{F} = |C|^2\}$ is normal (cf. [17]). If $\{F = C\}$ is normal, then also S is normal. Determine a homomorphism h , which we assume self-adjoint and such that h^2 is locally injective and finally

$$(5) \quad \begin{cases} h(F) = 0 \Rightarrow |F|^2 = |C|^2 \\ dh(F) = 0 \Rightarrow F = C \end{cases}$$

We can assume that $\{F = C\} \cap \{|F| = |C|\}$ is minimally defined. More precisely, if P is a singular point, we have that P can be reached through a regular and normal approximation from all directions. Particularly, $F(\phi) = A \int \phi d\sigma$ can be approximated by $|F|(\phi)$. For ϕ in a bounded set corresponding to the normal tube (cf. [16]), the corresponding operator $T_\Gamma(\phi)$ is normal.

We now claim that if the vorticity $w \in (\mathcal{B}^m)'(\Omega)$ and if $f, g \in \mathcal{B}^m(\bar{\Omega})$, where the monotropy-condition is on the boundary Γ to Ω , then if $\int_\Omega w_j(f, g)d\sigma \rightarrow \int_\Omega w(f, g)d\sigma$ through a normal and regular approximation and if $ww^* = w^*w$ and $\int |w|^2 d\sigma < \infty$, then the corresponding operator T_Γ is normal. If $\gamma_T = (f_T, h(f_T))$ is the regular approximation and if the dependence of T is polynomial in $\frac{d\gamma_T}{dT}$, according to Hurwitz theorem, since polynomials do not have zero's of infinite order, the zeroset for the polynomial must have measure zero. We can assume the parameter space in a domain of holomorphy or simpler, given existence of regular and normal approximations, we assume algebraic dependence of the parameter T in the tangent space, why all normal approximations algebraically dependent of the parameter T in the tangent space, can be assumed regular (by adding a regular approximation).

13. THE NIL-RADICAL WITH RESPECT TO $d\sigma$

As long as we are considering characteristic sets of measure zero, analytic continuation can be assumed. If $f^N \equiv 0$ on V such that $\sigma(V) = 0$, then we have seen that f^N has a polynomial representation. If $\sigma(V) > 0$, we have that $f \in \text{rad } I(V)$, but as $I(V)$ is a geometric ideal, we have $f \in I(V)$. Let

$$\mathcal{Z} = \{g \mid \int g d\sigma = 0\}$$

and $X = \{(x, y) \mid \int g d\sigma = 0 \mid g \in \mathcal{Z}\}$ and $\Omega = \{\zeta \mid \int g d\sigma = 0 \mid g \in \mathcal{Z}\}$. If $g = \text{const.}$ on a set V such that $dg = 0$ on V and if $g_j \rightarrow g$ in a normal and regular approximation, such that $dg_j \neq 0$ and $\int (g_j - g)d\sigma \rightarrow 0$ as $j \rightarrow \infty$ then we

see that $X = \{(x, y) \mid \int (g_j - g)d\sigma = 0 \quad j \rightarrow \infty\}$ is of measure zero. Assume that Ω is a domain of holomorphy and $f \neq 0$ on $\Omega \setminus V$, $\int_{\Omega \setminus V} f d\sigma = 0$, since $\int_{\Omega \setminus V} d\sigma = 0$ if $f^N \equiv 0$ in $\Omega \setminus V$, f^N has polynomial representation there. If $\mathcal{Z} \ni f_j \rightarrow f$ in a normal and regular approximation, then $f \in \mathcal{Z}$, so \mathcal{Z} is closed under normal convergence.

We can easily consider $(L_f) = \{g \mid \int_{(I)} (f - g)d\sigma = 0\}$, for f such that $\int_{(I)} f d\sigma < \infty$, so the case with finite area can be compared to the ideal \mathcal{Z} . We have earlier seen that $f^N \equiv 0$ on (I) implies $f = c$ on a set X . Let $X_1 = N(\{f \mid \int_{(I)} f d\sigma = 0\})$ and $X_2 = N(\{f \mid \int_{(I)} f^2 d\sigma = 0\})$ and so on. Then $\dots \subset X_N \subset \dots \subset X_2 \subset X_1$ and $X_1 = \cup_j X_j$

Lemma 13.1. *With the conditions above and if X_1 is a domain of holomorphy, we have $\sigma(X_j) \downarrow 0$ as $j \uparrow \infty$.*

It is easily seen that existence of a f such that $\int_{(I)} f^N d\sigma = 0$ and $f = \text{const} \neq 0$ on X_N implies $\int_{X_N} d\sigma = 0$. This gives a modified polynomial class

$$(6) \quad (R_\Omega) \quad \int_{\Omega} f d\sigma = 0 \Rightarrow \sigma(Z_f) = 0$$

If Ω is of measure zero, then holomorphic f are in (R_Ω) . If $\sigma(\Omega) > 0$, then polynomials are in (R_Ω) . For pseudo-differential operators, it may be more interesting to consider

$$(7) \quad (R'_\Omega) \quad \int_{\Omega} f(x+T) - f(x) d\sigma = 0 \Rightarrow \sigma(\Delta_{\mathbf{C}}) = 0$$

where $\Delta_{\mathbf{C}} = \{T \mid f(x+T) - f(x) = 0 \quad \forall x\}$.

We can also consider the class of finitely generated symbols

$$(8) \quad (FR'_\Omega) \quad \int_{\Omega} f^N(x+T) - f^N(x) d\sigma = 0 \Rightarrow \sigma(\Delta_{\mathbf{C},(N)}) = 0$$

where $\Delta_{\mathbf{C},(N)} = \{T \mid f^N(x+T) - f^N(x) = 0 \quad \forall x\}$. This defines, modulo monotropy, the class of symbols that become reduced with respect to the measure after finitely many iterations.

14. IDEALS OF HOLOMORPHY WITH INDUCED TOPOLOGY

It can be proved, that solvability for the second Cousin problem for an ideal of holomorphy (I) , depends on choice of topology for (I) . Assume (I) a geometric ideal and Ω a neighborhood of its zero's. For the first Cousin problem to be solvable we require that Ω is a holomorphically convex domain. For this domain, we have locally a pseudobase of holomorphic functions F_1, \dots, F_ν . We are interested in topologies for which this base can be selected as global. Such a topology is said to have Oka's property.

Given a covering of Ω , a ν -dimensional domain of holomorphy ($\nu > 1$) by open sets $\{\Omega_i \cap \Omega_j\}$ and associated continuous functions $g_{i,j}(\zeta) = e^{-A\rho_{i,j}(\zeta)}$ for a constant A , where $\rho_{i,j}$ a real or complex norm on Ω . Assume $|\rho_{i,j}| = 1$ and let $\rho_i(\zeta) = \zeta \chi_{\Omega_i}$, where $\chi_{\Omega_i} \in C_0^\infty(\Omega)$ are of modulus ≤ 1 and $= 1$ on Ω_i . Then

$g_{i,j}(\zeta) = e^{-A\rho_i\overline{p_j}}$. The topological space (Ω_i, g_i) has Oka's property and $\|G\|_{\rho,A} = \sup_{i,j} |g_{i,j}(\zeta)G(\zeta)|$ is a Banachspace over (I) .

The choice of $\rho_{i,j}$ can be varied, for instance if

$$(9) \quad c_1(1 + |\zeta|)^A \leq |G(\zeta)| \leq c_2(1 + |\zeta|)^B |\eta|^{-k}$$

where $\zeta = \xi + i\eta$ and $0 < |\eta| < r$ a positive constant r and where A, B are arbitrary constants. If G is analytic in the tube T^C , C an open, connected cone in \mathbf{R}^ν , then for a compact subcone C' , $b_{\Gamma}G$ exists uniquely in \mathcal{S}' . If

$$(10) \quad c'_1(1 + |\xi|)^{A'} \leq |G(\zeta)| \leq c'_2(1 + |\xi|)^{B'} |\eta|^{-k}$$

for constants A', B' and $\zeta = \xi + i\eta$, we have a topology which can be compared with (included in) the topology for invertible measures. In both these cases, Ω has Oka's property.

Assume the topology on Ω is defined through $g_i(\varphi)(\zeta) = \alpha I_{\widehat{F}}({}^tP\varphi)(\zeta)$ for a fixed $\varphi \in \mathcal{D}_{L^2}$. For $\zeta \in \Omega_i \cap \Omega_j$, the corresponding

$$g_{i,j}(\varphi)(\zeta) = \alpha\beta^{-1}I_{\widehat{F}}({}^t(P/Q)\varphi)(\zeta)$$

can be split into g_i, g_j as above. The corresponding topological space has Oka's property. Assume $(I)_{\widehat{\mathcal{D}}'_{L^2}}$ an ideal of holomorphy in the topology of

$\widehat{\mathcal{D}}'_{L^2}$, that is $f \in (I)_{\widehat{\mathcal{D}}'_{L^2}}$ means that $f = \sum_j P_j \widehat{F}_j$ for \widehat{F}_j reduced in both

variables separately and in $H \cap L^2$ and P_j polynomials. We can form the quotient ideal of holomorphy

$$(Q) = \{g \exists \phi \in H \ \phi g \in (I)_{\widehat{\mathcal{D}}'_{L^2}}\}.$$

If (Q) is equipped with $\widehat{\mathcal{D}}'_{L^2}$ -topology, ϕ can be chosen as polynomials. According to the theory of integral equations, given $g, F \in L^2$, there is a $\varphi \in L^2$, such that $g = I_{\widehat{F}}({}^tP\varphi)$, for a polynomial P . Assume Q' a polynomial such that $Q'I_{\widehat{F}} = I_{\widehat{F}}{}^tQ$, if ${}^tQ\psi \in L^2$ we have a $\varphi \in L^2$ such that ${}^tQ\psi = {}^tP\varphi$. This gives a representation of $g \in (Q)_{\widehat{\mathcal{D}}'_{L^2}}$. Note that a symbol in the ideal (I) equipped with this topology, will get a representation $f^\lambda(\zeta) = \sum_j I_{T_j}(\varphi)(\zeta)$ for $T_j \in (Q)_{\widehat{\mathcal{D}}'_{L^2}}$, for a $\varphi \in L^2$ and we can assume the support for φ adjusted to the corresponding subset in Ω .

15. THE HYPOELLIPTIC RADICAL FOR INDUCED TOPOLOGY

Assume now P a partially hypoelliptic, self-adjoint operator and $T = P(D)F$, where $F \in L^2(\mathbf{R}^\nu \times \mathbf{R}^\nu)$. Further,

$$\begin{aligned} \tau_h({}^tP\varphi)(\xi) &= {}^tP(\xi + h)\varphi(\xi + h) & {}^tP\varphi &\in L^2 \\ g_{h,P}(\varphi) &= \tau_h({}^tP\varphi) - {}^tP\tau_h\varphi & {}^tP\varphi &\in L^2 \end{aligned}$$

For a kernel F , such that $I_{\widehat{F}}$ reduced (left semi-bounded), since $I_{\widehat{F}}$ is a compact operator, if $\varphi \rightarrow 0$ in L^2 , then $I_{\widehat{F}}(\tau_h\varphi) \rightarrow 0$ and the reducedness implies $\tau_h\varphi \rightarrow 0$ in L^2 . Thus τ_h is a compact operator for this topology.

For test-functions ${}^tP\varphi \in L^2$, we have

$$I_{P\widehat{F}}(\tau_h\varphi) = I_{\widehat{F}}({}^tP\tau_h\varphi) = I_{\widehat{F}}(\tau_h({}^tP\varphi)) - I_{\widehat{F}}(g_{h,P}(\tau_h\varphi)) = I_1 + I_2$$

where we can assume $g_{h,P}$ is reduced. If $\varphi \rightarrow 0$ then ${}^tP\varphi \rightarrow 0$ in L^2 and we have $I_1 \rightarrow 0$. As $I_{P\hat{F}}(\tau_h\varphi) \rightarrow 0$, we have $I_2 \rightarrow 0$ and $\tau_h\varphi \rightarrow 0$. Thus τ_h is a compact operator also in this topology. Without the condition that $I_{\hat{F}}$ is reduced, we only have that translation is a weakly compact operator.

According to Malgrange (cf. [14] Theorem 2, Prel.) if ${}^t\tau : F' \rightarrow E'$ a linear continuous operator between (\mathcal{F}) -spaces such that $\tau(E)$ weakly closed in F then ${}^t\tau$ is a homomorphism and conversely. For the spaces where we have proved that the translation operator is weakly closed, we can apply the argument in section (cf. [7]) on $a_\Gamma = {}^t\tau$ and we see that the symbol p^N is reduced for the corresponding topology. We now use that $A_\lambda^N - P_\lambda^N = H_{\lambda,N}$ with $H_{\lambda,N}$ regularizing and assuming $A_\lambda^N \sim \text{Re } A_\lambda^N$ (equivalence of ideals), we see that A_λ^N is hypoelliptic with the initial topology. Note that for self-adjoint polynomial operators, $N = 2$ will do.

Let's return to our initial approach of $f = \text{Re } F$ in C^∞ and Exp . Except for lacunary cases, we assume that F^N is self-adjoint and hypoelliptic in an ideal (I_{HE}^*) , with induced topology of Schwartz-type. If the ideal of holomorphy is completely symmetric, that is $F \in (I_{HE}) \Leftrightarrow F^* \in (I_{HE})$, we can assume $F \sim F^*$, which implies $\text{Re } F^N \sim (\text{Re } F)^N \sim f^N$. More precisely, let

$$r(I_{HE}^*) = \{F \in H(\mathbf{C}^\nu) \mid F^N \in (I_{HE}^*) \quad \exists N\}.$$

That is entire functions for which the N 'th iterate corresponds to a self-adjoint and hypoelliptic operator. Let

$$S_r = \text{Re } r(I_{HE}^*).$$

The symbol f can naturally be compared with the (Hörmander-) class of symbols S^m , where we prefer to vary the complexification. For instance let $G = g + ih$ be a symbol and S_1^m the symbol-class with standard complexification. If $g \in S_r$ and $G^N \in S_1^{Nm}$, then $g \in S^m$ and $G \in S_1^m$. We have noted that the complexification that corresponds to operators hypoelliptic in \mathcal{D}' , S_2^m , must be different. Assume G corresponds to an operator hypoelliptic in \mathcal{D}' , $G^N = g_N + ih_N \in S_1^{Nm}$ for some N . If also $g \in S_r$, we must have $g^N \sim g_N \in S^{Nm}$. Assume existence of a $h' \in S^m$ such that $(h_N)^{1/N} \sim h'$ (a sufficient condition is that h_N is reduced), this defines $G \sim g + ih'$ (equivalence of ideals) in S_2^m .

Proposition 15.1. *The non-standard complexification of the symbol-ideals S_r , $S_r + iS_r^{1/N}$, produces (modulo \sim) the class of hypoelliptic operators.*

16. CONTRACTIONS OF FORMULAS

Assume (I) an ideal with Schwartz-type topology and a weakly compact translation. Consider for $\zeta \in \mathcal{B}$, a bounded domain in Ω pseudoconvex

$$\begin{array}{ccc} B(\zeta) & \longrightarrow & \alpha(T)B(\zeta) \\ \downarrow & & \downarrow \\ B(\zeta + T) & \longrightarrow & B(\zeta) + c_T \end{array}$$

If B is reduced, we have that $B(\zeta + T) = \alpha(T)B(\zeta)$, where $\alpha(T)$ is constant for $|T| > |T_0|$ and we see that B does not change sign in the T -infinity for $\zeta \in \mathcal{B}$. If $B = h(\psi) \in (I)$ with ψ reduced, for h algebraic, the earlier results can be

repeated. For $h = \tau_\epsilon w$ and w algebraic, the results follow modulo monotropy. Particularly, for ψ reduced, $\eta = h(\psi)/\psi$ does not change sign in the infinity.

Assume B reduced. Then the condition $B(\zeta + T)/B(\zeta) = 1 + c_T/B(\zeta)$, for $\zeta \in \mathcal{B}$ gives $c_T/B \sim_m 0$. Assume now that $dB(\zeta + T) = dB(\zeta) + dL_T(\zeta)$. We will put the condition that $dL_T/dB \sim_m 0$. It is sufficient through Rückert's theorem, if dB is reduced, that $dL_T \sim_m 0$. Consider

$$\Delta_{P/Q} = \{T \mid \frac{P}{Q}(\zeta + T) - \frac{P}{Q}(\zeta) = 0 \quad \zeta \in \mathcal{B}\}$$

(the set can without problem be extended to $\cup_{j=1}^\infty B_j$) with B_j as above)

$$(11) \quad \frac{-dL_T/dy}{dL_T/dx} = \frac{P}{Q}$$

Let's denote the set of T such that the last equality holds Ω_L . Thus $\Omega_L = \Delta_{P/Q}$. We say that L_T is reduced for contraction if

$$T \in \Omega_L \quad \frac{-dL_T/dy}{dL_T/dx} = \frac{dx}{dy} = \frac{P}{Q} \implies T = 0$$

We say that L_T is regular (for contraction) if for regular (x, y) and $T \in \Omega_L$, we have $dL_T/dT \neq 0$.

Assume $\alpha(T) = \psi(\zeta + T)/\psi(\zeta)$, for $\zeta \in \mathcal{B}$ and $\psi = e^\varphi$ reduced. Assume h algebraic and $\gamma(T) = h(\psi)(\zeta + T)/h(\psi)(\zeta)$ with $dh(\varphi) \sim_0 d\varphi$. Over Ω_L we have that (11) is obviously independent of T . As a consequence of Kohn ([13]) if $B(\zeta + T) = \gamma(T)B(\zeta)$ for B real and reduced, we have that γ is a polynomial in $T \Leftrightarrow L_T(\zeta)$ reduced in ζ with respect to lineality. Further, the set of T , such that γ not constant outside the origin is symmetric with respect to the origin and bounded. We will denote this set Ω_γ . For $0 \in \Omega_\gamma$, any regular approximation of the origin must go through Ω_γ . For $T \notin \Omega_L$ but close and $T \in \Omega_\gamma$

$$\frac{-dL_T/dy}{dL_T/dx} = \frac{-(\gamma(T) - 1)dB/dy}{\eta'_1(x) - \eta'(x)}$$

where $\eta'(x) = dh(x)(\zeta)/x$ and $\eta'_1(x) = dh(x)(\zeta + T)/x$. For h algebraic and such that $h^2 = 1$, we have that

$$\frac{dh(x)(\zeta + T)}{x} \sim_0 \frac{\alpha(T)dh(x)(\zeta)}{x}$$

Thus, over $\{y = B(\zeta)\}$ we have that $\frac{dL_T/dy}{dL_T/dx} \sim_0 \frac{\gamma-1}{\alpha-1}\eta'^{-1}(x)$. Outside Ω_γ , the quotient is bounded by a constant. We have $\frac{\gamma}{\alpha}(T) = \frac{\eta_1}{\eta}(x) \frac{x(\zeta)}{x(\zeta+T)} \sim_m \eta\left(\frac{\tau_T x}{x}\right)$.

Over the half-space $\{x \mid dh(x) \geq \mu x\}$, we know that η'^{-1} does not change sign, if $x \neq 0$. We have earlier seen that if η is bounded in the infinity, we have that $\eta_1 \sim_m 0$ and the associated ideal (with respect to h) has a global pseudo-base.

Assume now right-hand sides P, Q to the dynamical system such that P, Q becomes reduced by iteration. We can prove that $\Delta_{P/Q} \downarrow \{0\}$ for iteration. Define $L_{(2),T}(\zeta) = B^2(\zeta + T) - B^2(\zeta)$ for $\zeta \in \mathcal{B}$. We obviously have that $\Omega_{L_{(j),T}} \downarrow \{0\}$ as $j \uparrow \infty$. We can say that $L_{(j),T}$ is reduced for contraction. We can assume that on Ω_γ we have $dL_T/dT \neq 0$, if L_T is reduced for contraction. According to the

condition $\Omega_L \downarrow \{0\}$ for iteration, we can prove $\{(L_T - c_T) = \frac{d}{dT}L_T = 0\}$ is nowhere dense outside Ω_γ . Thus over $L_T = c_T$, we can assume L_T regular (for contraction).

Proposition 16.1. *Given a finitely generated system with polynomial right hand sides P, Q , assume that $\Delta_{P/Q}$, the lineality corresponding to P/Q , is $= \{0\}$. Then L_T is reduced for contraction. If P, Q are in the radical to reduced polynomials, we have $L_{T,(j)} = B^j(\zeta + T) - B^j(\zeta)$, becomes reduced for contraction, for j sufficiently large.*

17. PUISEUX-INDEX

Assume $(I)_S$ an ideal of holomorphy, with induced topology of Schwartz-type and with a reduced pseudo-base. Let $(I)_S^c$ denote elements developed over the pseudo-base in $(I)_S$ with constant coefficients and $(I)_S^h$ correspondingly with holomorphic coefficients. If $f \in (I)_S^c$ and $h(f) \sim_0 f$ (geometrically equivalent), then $h(f) \in (I)_S^h$. If also $h(f) \sim_\infty f$ (equivalent in strength), then $h(f) \in (I)_S^c$. Consider now the situation where f is reduced for lineality and $h(f), f$ are entire functions, such that locally $h(f) = \alpha f$, for $\alpha \in H(\Omega)$, where Ω is an open set in \mathbf{C}^ν . If h is locally injective, we have again the situation $h(f) \in (I)_S^c$, otherwise we assume h^N injective, for large N . Assume $N = 2$, we then have $h(\alpha)\beta$ constant, that is $h(\alpha) = C\beta^{-1}$. Define $\text{ord}_{0,L}F_1(\zeta) = k$, where $\zeta \in L$, L a line-segment through 0 and where the pseudo-base is assumed indexed after increasing order of zero's. If $k' = \text{ord}_{0,L}F_2(\zeta)$, we have the Puiseux-indexes for deviation $\text{dist}(A, C(A, 0)) \sim Cr^{k'/k}$ as $r \rightarrow 0$ and $|\zeta| \leq r$ [5].

Assume $h(f) = \alpha f$ with $\alpha \in H$ and h^N locally injective, for some N . Then

$$\frac{1}{(|1 - h|)}(f) = \left| \sum_j \alpha_j f + Cf \right| \leq \sum_j |\alpha_j| |f| + C_N |f|.$$

Thus $|f|^{-2} \leq 1 + C_N$, for a real constant C_N . If h^2 is injective and f reduced, for $\zeta \in Z_\alpha$, $1/f = f + c_2 f + \dots$, for constants c_2, \dots . Thus $f^2 = 1/(1 + C)$, for a f . If h is defined over J_h^ϵ , we can apply Proposition 10.1.

Assume (I) an ideal, not necessarily geometric and $(J) = \ker h$, for a homomorphism h , then $N(I \cap J) = \{\zeta \mid h(f)(\zeta) = 0 \mid f \in (I)\} = N_I(J)$, $N_{I_N}(J) = \{\zeta \mid h(f^N)(\zeta) = 0 \mid f \in (I)\}$. Assume h injective, for $N \geq N_0$, then $N_{I_N}(J) = N(I)$ for $N \geq N_0$. Thus, we have a division in components,

$$N_I(J) = N(I) \cup N_{I_2}(J) \cup N_{I_3} \cup \dots \cup N_{I_{N_0-1}}.$$

Further, $J_2 = I(N_{I_2}(J)) = \{g \in (I_2) \mid h(g^2) = 0\}$. We have seen that $V_c = \{\zeta \mid f(\zeta) = c\}$ and this means that $g \in (J_2) \implies V_c(g^2) \neq \emptyset$, for some c and for all $j < N_0$, the sets $V_c(g^j)$ are nontrivial for suitable constants c .

Assume now $f \in (I)_S^c$ and reduced and that h^2 is injective. Then,

$$|\zeta - z|^\sigma \leq |f(\zeta - z) - f(z)| + |f(z) - f(\zeta)| \leq |c| + \sum_k^\infty |f_\alpha| |\zeta|^{|\alpha|}$$

and if $|\zeta| < 1$,

$$|\zeta - z|^{\kappa+\sigma} \leq C + C_1 \sum_k^{\infty} |f_{\alpha}|,$$

where κ is dependent on the modulus of z . More precisely, assume $|z| < 1/R$, for $R > 1$. Then, since f is reduced, there is at least one α such that $1/R^{\kappa} < C_{\alpha} |f_{\alpha}(z)|$ and it is clear that κ depends on the value of R . Assume $|f_0| = \max_{|\alpha| \geq k} |f_{\alpha}|$ and $f_0 = \sum_j A_j F_j$ for $A_j \in H$, where the pseudobase is reduced and indexed by increasing order of 0 in z . Then,

$$f_0 = \sum_j A'_j F_1^{m_j} = (\sum_j A'_j F_1^{m_j - m_0}) F_1^{m_0}$$

according to the quantitative version of Rückert's Nullstellensatz (cf. [4]), we can find a m_0 such that $|\sum_j A'_j F_1^{m_j - m_0}| \leq C_j$ for all j . Thus, $|f_0|^{\frac{1}{\kappa+\sigma}} \leq C |F_1|^{\frac{m_0}{\kappa+\sigma}}$ and we can conclude, for V the zero's to $dh(f)$ over a set U_c , there are positive constants C_1, C_2, ρ, c and where $\rho < 1$

$$\text{dist}(\zeta, V) \leq |\zeta|^{\rho} (C_1 |F_1|^{\frac{m_0}{\kappa+\sigma}} + C_2)^{\rho}$$

Lemma 17.1. *If h is an algebraic homomorphism over an ideal (I) with $(I_N) = \{f^N \mid f \in (I)\}$ such that h is locally injective over (I_N) for $N \geq N_0$ and $(J) = \ker h$, then $N(J)$ is locally an algebraic variety.*

Note that if h is the extension-mapping defining the Schwartz-spaces, we have seen for our ideals (I) , that $r(I_{weak}) \sim I_{strong}$ and if h is algebraic and $(I') = h(I)$, $h(r(I_{weak})) \sim r(I'_{weak}) \sim (I'_{strong})$.

Lemma 17.2. *Assume X a reduced analytic variety and $I(X) = \ker h$ for a homomorphism h . Then h can be selected as algebraic.*

Proof:

Through the conditions that X is reduced we know that h can be selected as locally injective, thus within multiplication with a scalar we have that h is locally algebraic. \square

We have earlier studied symbols on the form $P(z) = e^{\frac{1}{z}} p_N^{\frac{1}{N}}$ with p_N reduced. We have then, for N suitable that $P(\xi) \rightarrow 0$ as $\xi \rightarrow -\infty$. In the same way for all derivatives to P , why $P(-z)$ has an infinite zero in the real infinity. In the same way, $Df(-z)$ has an infinite zero in the real infinity. These remarks can immediately extended to the case $n > 1$.

Lemma 17.3. *Assume p_N reduced in a neighborhood of a , V_a and $P \equiv 0$ in V_a , we then have $|P - e^{\frac{1}{a-z}} p_N(z-a)| < \epsilon$ on a $V \supset V_a$*

If a constant value is preserved for P in V we also have that $|P(z)| \leq Q(\frac{1}{|z-a|})$ in $V \setminus \{a\}$. We thus have negative indicator for P in a . Further using the relation in Lemma 17.3

$$e^{-C \frac{1}{|z|}} |P| \leq \epsilon e^{-C \frac{1}{|z|}} + |p_N(z)|$$

in a neighborhood of the origin. The corresponding result as $|z| \rightarrow \infty$ is that P is of type C .

Lemma 17.4. *Assume p_N as in Lemma 17.3 and $P^N \equiv 0$ on V_a , then we have*

$$|P - e^{\frac{1}{a-z}} p_N^{\frac{1}{N}}(z-a)| \leq \epsilon$$

Consider $I = \int_{\bar{\Gamma}_C} e^{\bar{z}} dz$ where the integral is taken along $\bar{\Gamma}_C$ clockwise. This can also be seen as $\int_{\Gamma_C} e^z dz$ where now the integral is taken in counter clockwise direction why 0 is considered as outer to Γ_C in the latter integral. Thus P can on the boundary to a cube with a finite distance to an infinite zero, be seen as a measure as in Lemma 17.3. If the infinite zero corresponds to P^N and not P , we have the representation as in Lemma 17.4.

In the case with the weighted lineality we define Γ as the boundary of a cube such that $\overline{\mathcal{F}} f$ has point support on Γ (finitely many) and $f = \text{const.}$ on Δ_x^Q . If $N(J) = \Delta_x^Q$ with (J) as before, we must have $I(\Delta_x^Q) = \text{rad}(J)$. Further $I(\Delta_x^Q)$ is a closed ideal. This means that if f is constant on the weighted lineality, then $\overline{\mathcal{F}} f$ has point support on Γ and conversely.

Proposition 17.5. *Assume μ_β are measures with point support on finitely many cubes and (J) the ideal as in section 18. Then we have $f \in \text{rad}(J) \Leftrightarrow \overline{\mathcal{F}} f = \sum_\beta D^\beta \mu_\beta$*

17.1. Estimates. Assume J_V^∞ denotes the ideal of symbols with an infinite zero on a set of positive complex dimension V . Consider phases on the form $h(\varphi) = P/Q$ with $P \in J_\infty$ (cf. section 33.6), where $J_\infty = \{g \mid D(f-g) = 0 \quad \exists f \in J_V^\infty\}$, that is, $g = f + c$ for a constant c in the infinity, Q is assumed hypoelliptic and self-adjoint. We know that for $zP(z) = F(\frac{1}{z})$, F has an infinite zero in the infinity ($z \rightarrow 0$). We also consider P such that $z^\rho P(z) = F_\rho(\frac{1}{z})$, for a real number $0 < \rho < 1$, where $F_\rho^{\frac{1}{\rho}}$ is assumed to have an infinite zero in the infinity. This would be the case for instance if $P^N \in J_\infty$. We define $I_\rho = \{P \mid z^\rho P(z) = F_\rho(\frac{1}{z}) \quad F_\rho^{\frac{1}{\rho}} \in J_V^\infty\}$ where as before V is a set of positive complex dimension containing the origin. If we let $V_r = \{z \mid |z| < r\}$ for $r < 1$ and $F'_\rho(\frac{1}{z}) = F_\rho(\frac{1}{z^\rho})$ then if I'_ρ corresponds to $F'_\rho \in J_{V_{r^{1/\rho}}}^\infty$, we have $I'_\rho = I_\rho$.

Lemma 17.6. *If $P \in I_\rho$ on V_a , a neighborhood of a , we have for p_N reduced on V_a that*

$$|P - e^{(a-z)^{-\rho}} p_N(z-a)| < \epsilon$$

and if $P^N \in I_\rho$ on V_a , that for $N \geq N_0$

$$|P - e^{(a-z)^{-\rho}} p_N^{\frac{1}{N}}(z-a)| < \epsilon \text{ on } V_a$$

Define a contour Γ according to $\Gamma = \{\zeta \mid h(e^\varphi)(\zeta) = 0\}$ and the neighborhood $\Gamma_\epsilon = \{\zeta \mid |h(e^\varphi)(\zeta)| < \epsilon\}$. The Rückert Nullstellensatz now gives that

$$|\lambda| = |e^{h(\varphi)}/Q| < \epsilon M$$

on a poly-disc neighborhood of Γ . Define the ideal (I_Γ) so that $N(I_\Gamma) = \Gamma$. Let $I_\Gamma(u) = \int_\Gamma e^\varphi u dx$. Then there exists a homomorphism h_1 such that $I_\Gamma(h_1(u)) = 0$.

Note that for $\overline{(I_\Gamma)}$ with $e^\varphi \in (I_\Gamma)$ the closure taken in Whitney-norm we have that $dh(e^\varphi) = 0$ on Γ and $e^\varphi dh_1 = 0$ on Γ , for $e^\varphi \in \overline{(I_\Gamma)}$.

If h is not reduced, we denote by $I_{\Gamma,h}$ the integral $\int_\Gamma e^{h(\varphi)} u dx$ and by Rückert's Nullstellensatz we can realize $I_{\Gamma,h}$ as a measure in Γ_ϵ $| I_{\Gamma,h}(u) | \leq C_\Gamma M \epsilon \sup_\Gamma | u |$. We consider particularly neighborhoods of Γ such as $U_c = \{ \zeta \mid |\zeta|^c d(\zeta, \Gamma) < C \}$ for positive constants c, C and the associated sets $U'_c = U_c \setminus \Gamma$. If $h(\varphi) \in J_\infty$ there are $g \in J_\infty^\varphi$ such that $D(h(\varphi) - g) = 0$ on U_c and $h(\varphi) = g + a$ for a constant a . We also consider $h(\varphi) = f + g + a$ where g and a are as before and f has negative order $-\rho$, for $0 < \rho < 1$. Immediately $d(\zeta, \Gamma)$ has negative order on U_c and by studying $1/(\zeta - \zeta_0)$ where $\zeta_0 \in \Gamma$ we can compare with indicator theory and we see on U'_c that $\limsup \frac{1}{r} \log |d| = -\tilde{\rho}$ where $\tilde{\rho}$ is close to 1. We know that if V is an algebraic set, then U_c is a domain of holomorphy. Assume h such that h^N is locally injective for $N \geq N_0$, we have a decomposition of U_c according to $1 = P_k + R_k$, for $k < N_0$ with P_k polynomial and R_k reduced. We have the same decomposition if the set $\{h^N(e^\varphi) = \tilde{c}h^N(e^\varphi)\} \uparrow U_c$ as $N \uparrow \infty$. Assume $\zeta \in \Gamma \cap V = \Gamma \cap E^c$ (the complementary set to the polar set to $h(\varphi)$), then $\tilde{h}(e^\varphi)$ is a polynomial in a neighborhood of ζ . The decomposition of U_c gives a decomposition of the distance-function $d(\zeta, \Gamma) = d_R(\zeta, E) + d_V(\zeta, \Gamma)$ where the terms are extended by zero on the respective complementary set. Note that d_V defines a norm. Using an a-priori estimate by Palamodov (cf. [20]) we have

$$\log |d| - c \leq h(\varphi) \leq \log |d| + c$$

and by taking limes sup. as $r \rightarrow \infty$ we get

$$-\tilde{\rho} - c \leq \limsup \frac{1}{r} h(\varphi) \leq -\tilde{\rho} + c$$

Thus for a sufficiently small neighborhood U'_c , $e^{h(\varphi)}$ has negative indicator. If h is such that h^N is locally injective for $N \geq N_0$ we see that also U_c is decreasing by iteration as is V_N . More precisely we can write $U_{c/N} \downarrow$ as $N \uparrow \infty$ and

$$-\tilde{\rho} - c/N \leq \limsup \frac{1}{r} h^N(\varphi) \leq -\tilde{\rho} + c/N$$

and for a sufficiently large iteration index, we have that $h^N(\varphi)$ has negative order.

18. TRANSVERSALITY

Assume T is a compact operator and of type 0 over a (reduced) ideal (I) . Then, $S(f) = \frac{T(f)}{f}$ is of negative type over (I) (of type $-\infty$). Let \mathcal{K}

$= \{g \mid T(g) - R(g) \text{ of type } 0 \mid R \text{ of type } -\infty\}$. For $g \in \mathcal{K}$, we have $\frac{T(g)}{g}$ is of type 0 and also of type $-\infty$. Assume, $T^N - I \sim_m 0$. This means that there is an entire function R , such that modulo monotropy, $T^N - I = R$ in Exp_0 .

Lemma 18.1. *Assume R entire and $R \sim_m 0$, then there exists a R_1 of type $-\infty$ such that $R - R_1$ is of type 0.*

This means that for K a compact operator, such that $K \sim_m 0$, we have $\mathcal{K} = (I)$ and for a compact operator such that $K^N \sim_m 0$, from some $N \geq N_0$, we have that $\mathcal{K} = \text{rad}(I)$.

18.1. Transversality modulo monotropy. Assume $h(g) = w\mathcal{F}(g) - \mathcal{F}w(g)$ for $g \in J$ and given a homomorphism w , such that w is locally factorized by $\tau_\epsilon v$, for an algebraic homomorphism v .

Lemma 18.2. *For any $g \in (I)$ an ideal in H and a homomorphism w defined on (I) , there is a $g_1 \in H$ such that $w(g) = \tilde{w}(g_1) = I_{[w, I]}(g_1)$*

Thus $g \in (I)$ can be realized through $Id \in H'$ and $Id(g_1) = g$. The composition of analytic functionals, here realized through measures, is given as convolution of measures. $Id(\mu)(g_1) = \int_V g_1 d\mu$ and $Id(\mu)^2(g_1) = Id(\mu * \mu)(g_1)$. We write $g^2 = g_1 * g_1$ for short. Let $h'(g^2) = v\mathcal{F} g^2 - \mathcal{F} v(g^2) = v^2 \mathcal{F}(g_1) - \mathcal{F} v^2(g)$. Further, we note that $h'^2(g) = (v\mathcal{F} - \mathcal{F}v)^2(g) = (\mathcal{F}v)^2(g) - v\mathcal{F}\mathcal{F}(g)$.

Lemma 18.3. *For any $g \in (I)$ and a homomorphism v acting on (I) , there exists a g_0 such that $\mathcal{F} v^2(g_0) = -(\mathcal{F}v)^2(g)$.*

The proof is analogous to the proof of the previous Lemma. Finally, assuming as above that $v\mathcal{F}(g) = \mathcal{F}v(g) + c$, for a constant c . If $(\mathcal{F}g_1)^2 = \mathcal{F}(\mu * \mu)(g) = \mu * \mu(\mathcal{F}(g_0)) + c$, where μ corresponds to the functional Id , thus there is a g_2 such that $-\mathcal{F}\mathcal{F}v(g) = v\mathcal{F}g_2 + c$, for a constant c . Thus, we could say that $d(h'(g^2) - h'^2(g)) = 0$ and that h' modulo monotropy and within representation in H' is an algebraic homomorphism in the tangent space.

18.2. A lifting principle. We can define a vector-field along a morphism h according to

$$\begin{array}{ccc} X & \xrightarrow{\phi} & T(Y) \\ h \searrow & & \downarrow \pi \\ & & Y \end{array}$$

We then know that $\phi^* = \phi$ is equivalent to the proposition that $dh = h d$. If this is the case we have for our application that $d\psi$ and ψ are not in the respective "eigen-spaces" $(I_\mu), (I_\mu^\perp)$ on the domain Ω .

Consider the intersection between the transversals and the leaves to the foliation (modulo monotropy). We know that if

$$(12) \quad P/Q - y/x$$

changes sign in a domain Ω , then this intersection is not discrete (that is a characteristic γ to the system runs partially in this intersection). The proposition is thus that there is a ψ non-constant such that

$$\frac{h(\psi)}{\psi} = \mu \quad \frac{dh(\psi)}{d\psi} = \frac{1}{\mu}$$

If h is such that h^2 is locally injective (modulo monotropy algebraic), we can write (for all j) $\frac{d}{d\zeta_j}(h^2(\psi) - \psi^2) = 0$ or locally $\frac{d}{d\zeta_j}(c\psi - \psi^2) = 0$, but then ψ would be constant in contradiction with the conditions. We conclude that

Proposition 18.4. *Given a homomorphism h algebraic modulo monotropy and such that h^2 is locally injective then the intersection between the transversals and the leaves to the foliation is discrete.*

Fredholm operators. Differential operators have a natural interpretation as Fredholm-operators. We consider the standard projections $P : L^2 \rightarrow R(L_\lambda)$, $Q : H^{s,t} \rightarrow N(L_\lambda)$, where $H^{s,t}$ is the Sobolev-space with tensorized weights and L_λ the differential operator $L(D) - \lambda$, for a real or complex parameter λ . $L_\lambda \in \Phi(H^{s,t}, L^2)$ (Φ denotes the class of Fredholm operators) gives a decomposition

$$H^{s,t} = X_0 \oplus N(L_\lambda) \text{ and } L^2 = Y_0 \oplus R(L_\lambda),$$

where $N(L_\lambda)$ denotes the solutions to the homogeneous equation and $R(L_\lambda)$ the range of L_λ . It is not difficult to prove that, for $\sigma > 0$, L_λ is homogeneously L^2 -hypoelliptic, which means that $N(L_\lambda) = Q(H^{s,t}) \subset C^\infty$, that is Q is regularizing on $H^{s,t}$ and E_λ is a left parametrix to L_λ . If L_λ is homogeneously L^2 -hypoelliptic, then also its Hilbert-space adjoint is homogeneously L^2 -hypoelliptic, that is $P^\perp(L^2) = N(L_\lambda^*) \subset C^\infty$, and P^\perp is regularizing on L^2 . We conclude that E_λ is a left and right parametrix to the operator L_λ . Also, if σ arbitrary, that is L_λ not necessarily homogeneously L^2 -hypoelliptic, we can extend the definition of E_λ with a regularizing term and we get a hypoelliptic action on, at least part of L^2 . If $E_\lambda \in \Phi$, is on the form $I - K$, then there is a positive integer N_0 such that $N(E_\lambda^N) = N(E_\lambda^{N_0})$ for all $N \geq N_0$. Since adding a compact operator to the Fredholm-inverse, does not change the index or the form $I - K$, we let E_λ be defined as

$$\begin{cases} L_\lambda^{-1} + X \text{ on } R(L_\lambda) \text{ with } X \in C^\infty \\ X' \in C^\infty \text{ on } R(L_\lambda)^\perp \cap N(E_\lambda^M)^\perp \text{ for a suitable } M \\ f \in L^2 \text{ otherwise.} \end{cases}$$

Thus, this operator L_λ is hypoelliptic on $N(E_\lambda^M)^\perp$.

19. ANALYTIC HYPOELLIPTICITY IN L^2 FOR THE ITERATED OPERATOR

We shall in this section only consider constant coefficients, differential operators, with D_x replaced by (iD_x) . Consider the translation operator on $H(E_{\mathbf{R}})$, $a_\Gamma u(\xi_x) = u(\xi_x + it\eta_0)$, for $t\eta_0 \subset \Gamma$. Let

$$A_\Gamma = \mathcal{F}^{-1} a_\Gamma \mathcal{F} : H(E_{\mathbf{R}}) \rightarrow H'(E_{\mathbf{R}}).$$

Here, \mathcal{F} denotes the Fourier-Borel transform and Γ some simple cone in $\Delta_{\mathbf{C}}$ and $E_{\mathbf{R}}$ denotes in this context some subset of \mathbf{R}^ν or the entire \mathbf{R}^ν . If $\eta_0 \in \Delta_{\mathbf{C}}(P)$, we have $P(iD)A_\Gamma u = A_\Gamma P(iD)u$. Consider the real quotient homomorphism, $\varphi : E_{\mathbf{R}} \rightarrow E_{\mathbf{R}}/i\Delta_{\mathbf{C}}$. Let $a_{\varphi(\Gamma)} u(\xi_x) = u(\varphi(\xi_x))$, we then have $b_{\varphi(\Gamma)} = \lim_{t \rightarrow +0} a_{\varphi(\Gamma)}$. We define $B_\Gamma = \lim_{t \rightarrow +0} A_\Gamma$. The definition of b_Γ (cf. [21]), is not dependent on the neighborhood of 0 when we argue that this means that b_Γ is quasi-portable by $E_{\mathbf{R}} \times i0$. The same arguments give that $b_{\varphi(\Gamma)}$ is quasi-portable by $E_{\mathbf{R}} \times i\Delta_{\mathbf{C}}$.

For $u \in L^2$, we can make a composition, $u = u_+ + u_-$, where \widehat{u}_- , has support contained in cones with indicator $\delta_-(\eta_0) < 0$. Note that the indicator for a cone Γ is here defined as $\delta(\eta_0) = \sup_{y \in \Gamma} \langle y, \eta_0 \rangle$. According to ([21]), u_- can be considered as regular elements and it is sufficient to consider $u_+ = u - u_-$. We

argue that, to conclude that the operator $P(iD)$ is analytically hypoelliptic on L^2 , it is sufficient to study the set of complex lineality and the action on u_+ - terms.

Assume $h_F(\eta)$ the indicator of growth, corresponding to B_Γ . Using [15](Ch.2, Theorem 4.3), it can be shown, that for $\eta \in \Delta_0 = \Delta_{\mathbf{C}} \setminus 0$, $h_F = g$, where g is the indicator of growth corresponding to the frequency component Σ in $\text{WF}_A(u)$. Thus any cone in Δ_0 has indicator ≥ 0 . More precisely, let W denote the convex hull of the real support for B_Γ , that is $W = \{y; < y, \eta > \leq h_F(\eta) \mid \|\eta\| = 1\}$. Let

$$W_+ = \{y \in W; < y, \eta > \geq 0 \mid \|\eta\| = 1\}$$

and let W_- be the complementary set. Let

$$V_+ = \{\eta; < y, \eta > \geq 0 \mid y \in W_+\}.$$

Further, $\Delta_0 \cap V_+ = \{\eta; < y, \eta > = 0 \mid y \in W_+\}$. Since $g(\eta) = h_F(\eta) = 0$ over $\eta \in \Delta_0$, we must have $\Sigma \cap \Delta_0 \cap V_- = \emptyset$. Next, using [15] (Ch.2, Prop. 5.2) and conjugated indicators, we can prove a support theorem in $H'(E_{\mathbf{R}})$, that is

Proposition 19.1. *If $P_\lambda(iD)$ is a constant coefficients differential operator and $u \in H'(E_{\mathbf{R}})$, $P(iD)u \in L^2(E_{\mathbf{R}})$, we have in H' , $\text{supp } P(iD)u = \text{supp } u$.*

Let Σ be such that $b_\Gamma \neq 0$, Σ' the frequency component corresponding to $\text{WF}_A(P_\lambda u)$. Knowing that $\Sigma' \subset \Sigma$ and $\Delta_0 \subset \Sigma \setminus \Sigma'$ (that is a necessary condition for hypoellipticity is that the operator is reduced for complex lineality), we have

$$\Sigma = \Delta_0 \cup \Sigma' \setminus \Delta_0 \text{ and } \Delta_{\mathbf{C}} \cap \Sigma' = \{0\}$$

and the lineality can be regarded as the "boundary" for Σ . We claim that cones Γ_j exist, such that $\Delta_0 \subset \cup_j \Gamma_j \subset \Sigma$. If we assume the operator P_λ , such that

$\text{Re } P_\lambda \sim P_\lambda$, it will be sufficient to study the real part of the polynomial, which is considered as a real analytic function. This means that a proposition (cf. [15] ch2, sec.4, Cor.2) can be applied on B_Γ which gives that if B_Γ is quasi-portable by $E_{\mathbf{R}} \times i\Delta_{\mathbf{C}}$, then it is portable by the same set. Thus, we get a corresponding representation $B_\Gamma = \sum_j B_{\Gamma_j}$. We also note that b_Γ is in L^2 strictly portable by $E_{\mathbf{R}} \times i\Delta_{\mathbf{C}}$, because of the properties of the restriction-mapping in this space.

Lemma 19.2. *Assume P_λ with constant coefficients and self-adjoint and that P_λ is reduced with respect to the set of lineality, $\Delta_{\mathbf{C}}$, $P_\lambda u \in L^2$ for some distribution u and such that $\text{WF}_A(P_\lambda u) = \emptyset$. Then $b_\Gamma(u)$ is strictly portable by open connected cones with indicator < 0 .*

Proof: Assume contrarily, that L_+ is a line in Γ_+ , an open connected cone with indicator ≥ 0 and that $b_{L_+}(u) \neq 0$. We can write

$$(13) \quad P_\lambda b_{L_+}(u) = b_{L_+}(P_\lambda u) + C_{P_\lambda, L_+}(u)$$

where $C_{P_\lambda, L_+}(u) = [P_\lambda b_{L_+} - b_{L_+} P_\lambda](u)$. We can assume that u is such that $\hat{u} \neq 0$ on L_+ . We have either $C_{P_\lambda, L_+}(u) = 0$, in which case the reducedness condition implies $L_+ = 0$ or $C_{P_\lambda, L_+}(u) \neq 0$, which contradicts that $\Delta_{\mathbf{C}} \cap \Sigma' = \{0\}$. \square

Consider the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{C}_X$, which assigns to sections $s \in \mathcal{O}_X(U)$, their continuous functions $[s] \in \mathcal{C}_X(U)$. According to [9], as a consequence of the Rückert Nullstellensatz, the kernel to this homomorphism, is

the nilradical \mathcal{N}_X . Using [15] and [9], the homomorphism can be regarded as an evaluation functional on H_U . That is $E(\varphi)(x) = \varphi(x)$, for $\varphi \in H_U$. Using the nuclearity of H'_U , we could write $I_E(\varphi)(x) = \varphi(x)$. The cited result means that $I_E^N(\varphi)(x) = 0$, for N large and $\varphi \in H_U$ and iteration in this context corresponds to composition of convolution kernels. More precisely, let $\Delta \in H'(\mathbf{C}^\nu \times \mathbf{C}^\nu)$, be the kernel representing Id_x ($x \neq 0$). It follows from the argument below, that in a neighborhood of x , V , we have for this kernel,

$$\langle I_\Delta^N(\varphi), \psi \rangle_V = \langle I_{[\Delta, \Delta]}^N(\varphi), \psi \rangle_V = 0, \text{ with } \varphi, \psi \in H(\mathbf{C}^\nu), \text{ for some } N.$$

Define an ideal \mathcal{J} , as holomorphic functions, invariant for complex translation with $t\eta$, $\eta \in \Delta_{\mathbf{C}}$ and $t\eta \in \Gamma$, that is $\mathcal{J} = \{\varphi \in \mathcal{O}(U) ; a_\Gamma \varphi = \varphi\}$. We then have

$$V_1 = U_1 \setminus \Delta_{\mathbf{C}} \subset N(\mathcal{J}) = U_1 = N(\mathcal{N}_{U_1}).$$

The construction can be iterated, $\mathcal{J}^2 = \{\varphi \in \mathcal{O} ; a_\Gamma \varphi^2 = \varphi^2\}$, with $V_2 = U_2 \setminus \Delta_{\mathbf{C},2} \subset U_2$, where $\Delta_{\mathbf{C},2}$ is the lineality corresponding to the operator P^2 . Note that since $U_j = V_j \cup \Delta_{\mathbf{C},j}$, we have $\mathcal{J}(U) = J_1 J_2$ and $J_2 = \{\varphi \mid a_\Gamma \varphi = \varphi\}$ $N(J_2) = V_j$. Thus, as a_Γ is a homomorphism, we have that if $\Delta_{\mathbf{C},j} \downarrow \{0\}$, then also V_j are decreasing, as the index grows. The sequence U_N , constructed in this way, is thus for a partially hypoelliptic operator, decreasing (with respect to inclusion of sets) and $0 \notin V_N$, for all N , but $P \in \mathcal{J}_0^N$, for all N . From the construction,

$$N(\mathcal{J}^N) = N(\mathcal{N}_{U_N}) \text{ which implies } \text{rad } \mathcal{J}^N = \mathcal{N}_{U_N}, \text{ for all } N.$$

Using that $\Delta_{\mathbf{C},N}$ is an analytic set, for U_N sufficiently small, the corresponding nilradical will be the zero ideal. The conclusion must be that $P \in \mathcal{J}_t^N$ implies that $t = 0$ and the iterated operator P^N , is in this sense reduced.

Proposition 19.3. *In the case of several dimensional translations, (product topology) we have that if P is a partially hypoelliptic operator with constant coefficients, then there is a N_0 such that for $N \geq N_0$, P^N is reduced for complex lineality.*

20. b_Γ CONSIDERED ON $\text{Exp}_{\|\cdot\|,A}$

Assume, that for a holomorphic function g

$$(14) \quad |g(\zeta)| \leq C e^{B|\text{Im } \zeta| + A|\text{Re } \zeta|} \quad \zeta \in \mathbf{C}^\nu$$

for a positive constant C . If the real and positive numbers A, B can be chosen arbitrarily small, the corresponding operator allows real support. According to [15], if g is entire in \mathbf{C}^ν and $|g(iy)| \leq C(\epsilon) e^{\epsilon\|y\|}$, $|g(x)| \leq C'(\epsilon) e^{(A+\epsilon)\|x\|}$, for $x, y \in \mathbf{R}^\nu$, then $|g(x + iy)| \leq C''(\epsilon) e^{(A+\epsilon)\|x+iy\|}$, where A is positive number, determined by the indicator, corresponding to g and $\|\cdot\|$ is a norm in \mathbf{C}^ν . This means that it is possible to consider b_Γ in $\text{Exp}_{\|\cdot\|,A}$. The following limits can be determined.

$$\begin{aligned} \lim_{y \rightarrow 0} [e^{-(A+\epsilon)\|x+iy\|} g(x + iy) - e^{-(A+\epsilon)\|x+iy\|} g(x)] \\ \lim_{y \rightarrow 0} [e^{-(A+\epsilon)\|x+iy\|} g(x) - e^{-(A+\epsilon)\|x\|} g(x)] \end{aligned}$$

which means, (cf. the Lemma below), that

$$b_{\Gamma}[e^{-(A+\epsilon)\|x\|}g(x)] = e^{-(A+\epsilon)\|x\|}g(x) \text{ in } \text{Exp}_{\|\cdot\|,0}$$

The corresponding operator can now be considered as portable by a ball of radius A with respect to some complex norm or by a tube $\{x \in \mathbf{R}^{\nu}, |x_j| < A + \epsilon'\} + i\mathbf{R}^{\nu}$

Lemma 20.1. *Given an entire function g of finite type A , the entire function $g(\zeta) - g(x)$, $\zeta = x + iy$, is of (complex) type 0, that is, for all $\zeta \in B_{\epsilon} + i\mathbf{R}^{\nu}$, there is a positive constant C ,*

$$|e^{-\epsilon\|\zeta\|}[g(\zeta) - g(x)]| \leq C$$

Proof: Let $M_{\alpha}(t) = \sup_{\|\eta\| \leq t} |g_{\alpha}(i\eta)|$. In general, if g_{α} is entire for all α (analogous to Weierstrass M-test) we have $M_{\alpha}^{1/|\alpha|}(t) \rightarrow 0$ as $|\alpha| \rightarrow \infty$, for every t . This means that $\sum_{\alpha} g_{\alpha}(i\eta)x^{\alpha}$ is an entire function of $\zeta = x + i\eta$. Let

$$\Gamma_g = \{\eta, e^{-\epsilon\|\eta\|}[g(x + i\eta) - g(x)] = 0 \ \forall x \ \forall t\} = \Delta_{\mathbf{C}}(g)$$

Then, for $\eta \in \Gamma_g$, $g_{\alpha}(i\eta) = C_{\alpha}$, a constant independent of t . Thus $M_{\alpha}^{1/|\alpha|}(t) \rightarrow 0$ over Γ_g as $|\alpha| \rightarrow \infty$, for all t . Assume η close to $\eta_0 \in \Gamma_g$. Then, $g_{\alpha}(i\eta) = \sum_{\beta} g_{\alpha\beta}(i\eta_0)(it(\eta - \eta_0))^{\beta}$. Now, to see that g_{α} is of exponential type 0, let $M_{\alpha\beta}(t) = \sup_{\|\eta\| \leq 1} |g_{\alpha\beta}(i\eta)|$ and consider

$$\frac{\log M_{\alpha\beta}(t)}{t} \leq \epsilon |\alpha| \quad \forall \alpha$$

The left side has the indicator for $g_{\alpha\beta}$ as limes superior, as $t \rightarrow \infty$. As $|\beta| \rightarrow \infty$, the limit on the left side, times $1/|\beta|$, is 0, for all $t > 1$. Thus, g_{α} is holomorphic and of type 0, in any conical neighborhood of Γ_g . \square .

This may be regarded as a generalization of the proposition that some iterate of the polynomial defining the lineality to a symmetric operator, is a hypoelliptic polynomial. Given g of type A , we then have that $e^{-A\|x\|}b_{\Gamma}(g) \in \text{Exp}_{\|\cdot\|,0}$.

Concerning self-adjointness, the following propositions are immediate.

Proposition 20.2. *Assume $b_{\Gamma_+}f = g_A^+ \in \text{Exp}_{\|\cdot\|,A}$. A sufficient condition for self-adjointness of I_E is that $2E = g_A^+ + \overline{g_A^+}$. A necessary condition, is that $\text{Im } g_A^+ = b_{\Gamma_-} \text{Im } f$.*

It is a consequence of the previous argument, that if f is of type 0, then the corresponding operator will be self-adjoint in Exp . It is also a consequence, that hypoelliptic operators can be considered as of (complex) type 0 (consider for instance them as acting on parametrices to hypoelliptic polynomial operators).

21. SOME REMARKS ON WEYL-CALCULUS

In any theory involving the strong Fourier-transform, particular care has to be taken with the sets V_c . Consider for instance Weyl-calculus. The fundamental relations are

$$a(x, \xi) = \int K(x + t/2, x - t/2) e^{-i\langle t, \xi \rangle} dt$$

$$K(x+t/2, x-t/2) = \frac{1}{(2\pi)^n} \int a(x, \xi) e^{i\langle t, \xi \rangle} d\xi$$

where $a \in S^m$ and the operator $a(x, D): \mathcal{S} \rightarrow \mathcal{S}$. Assume $a(x, t) = a(x, 0)$ for t on a line. Then, $K(x+t/2, x-t/2) = K(x, x)$ or $\tau_{t/2} I_K \tau_{t/2} = I_K$. This situation may occur if $h(a)(\zeta) = 0$, for a radically injective homomorphism h and $N(I_h)$ corresponds to a generalization of the lineality. On these lines $K(x, x) = c\delta_0$ (the Dirac-measure) and $D_t^\alpha a(x, t)$ is a polynomial, for x fixed.

Lemma 21.1. *Assume F a reduced kernel in $L^2 \cap H$, then $\mathcal{F}^{-1} F \in L^2 \cap H$*

Otherwise, if $X = X' \setminus \Gamma$ reduced, for a line Γ and $X' \subset \mathbf{R}^n$, we have $\mathcal{F}^{-1} F = c\delta_\Gamma + \mathcal{F}^{-1}[F|_X]$, that is $\mathcal{F}^{-1}[F - I] \in L^2 \cap H$. Assume now, $a(x, \xi) = \sum_j a_j f_j = \sum_j I_{[A_j, F_j]}$ over $L^2 \cap H$, with F_j reduced in both ξ -variables separately. Then $\mathcal{F}^{-1} a(x, \xi) = \sum_j I_{[A_j, \mathcal{F}^{-1} F_j]}$ and $[A_j, \mathcal{F}^{-1} F_j] \in L^2 \cap H$. Further, $I_K = \sum_j I_{[\tau_{-t/2} A_j \tau_{-t/2}, \mathcal{F}^{-1} F_j]}$, that is

$$\frac{1}{(2\pi)^n} \int \int A(x-t/2, y+t/2) F(y, \xi) e^{i\langle t, \xi \rangle} d\xi dy$$

If A_j is constant on $t/2$ -lines, then $I_K = \sum_j c_j I_{[\delta, F_j]}$. This phenomena does not occur for reduced symbols (the coefficients reduced), since the sets V_c then reduce to points. If thus $a \in \overline{\mathcal{T}}_h$ and $a \sim_0 \sum_j a_j f_j$, with f_j reduced and a_j constants, we have,

$$\tau_{t/2} I_K \tau_{t/2} = \sum_j a_j I_{\mathcal{F}^{-1} F_j} = \mathcal{F}^{-1} a$$

with $\mathcal{F}^{-1} F_j \in L^2 \cap H$. Note again that the sum does not have to be reduced. Also, note that in analogy with earlier results, we have "locally" a polynomial representation $a_t^w - p_t = H_t$, with H_t regularizing and p_t polynomial.

22. LINK BETWEEN THE DYNAMICAL SYSTEM AND THE WAVEFRONT SET

Given existence of ψ not constant such that $h(\psi) = \mu\psi$ and if also $Q = \mu P$ for a constant μ and where P, Q are polynomial right hand sides to the system under consideration, we have that $H_h(\psi) = 0$ why H_h has a real integral curve. Assume $W_c = \{(\eta_1(\zeta), \zeta) \mid \zeta \in V_c\}$ where V_c is the foliation corresponding to the symbol to a differential operator with analytic coefficients and $\eta_1(\zeta)$ is a real-analytic function (assume order one). Assume Γ a tangent-line to the foliation (restriction to one variable) and $\Gamma \rightarrow L$ mapping dual line-segments in $Exp_{\rho, A}$ with respect to the Fourier-Borel transform. Let W_Γ be the corresponding set $\{(\eta_1(\zeta), \zeta) \mid \zeta \in \Gamma\}$. For the situation with $\zeta \in V_c$ there are corresponding functions $\psi \in (I_c)$ and a corresponding tangent equation $Q - cP = 0$ where P, Q are assumed polynomials. Let t be the mapping $(I_c) \rightarrow \Gamma$, where we can obviously assume that singular points to the system are mapped on foliation points and that regular points are mapped on the outside of the foliation. Further, for solutions u to the homogeneous equation to the differential operator above, it is not difficult to find a mapping $v: W_\Gamma \rightarrow W F_a(u)$. Denote with $\tilde{\gamma}$ the image of γ under the mapping

$\gamma \rightarrow W_\Gamma \rightarrow WF_a(u)$, then according to Hanges' theorem (cf. [21]), there is a neighborhood of the origin U such that $\tilde{\gamma}(U) \subset WF_a(u)$.

Starting with a complex line Γ , we consider the transversal L as linearly independent. Γ corresponds as before to an ideal (I_{μ_1}) , for a real number μ_1 . Let $(I_{\mu_1})'$ be the ideal corresponding to the transversal L . In a neighborhood of a singular point to the associated dynamical system, we can in this way construct a symplectic base. If $(I) = \bigoplus_\mu (I_\mu)$ (finite sum) and if η is cyclic we have thus $\eta(\psi) = \phi$ for a $\phi \in (I_{\mu_1})'$, that is η maps tangents on transversals and we could claim that the transversal is locally unique. Note that in the case where $\psi = e^\varphi$ if also $\frac{d}{d\varphi}h = h\frac{d}{d\varphi}$, then $(I_\mu) = (I_\mu)'$, but we are still assuming polynomial right hand sides for the associated dynamical system, why there are only finitely many tangents and we will denote half of them transversals and also in this case we can construct a local symplectic base. More precisely assume h algebraic such that $h(g^2) = h(g)^2$ for g real and $h^* = h$. The corresponding eigen-values must be real and positive, that is $h(g^2) \geq 0$ and $h(g^2) = cg^2$ with c real and non-negative. Assume $c = \mu^2$ for μ real, then the "eigen-spaces" can be seen as linearly independent $(I_\mu)_+ \oplus (I_\mu)_-$. We can set $(I_\mu)' = (I_\mu)_+$. This means that η maps tangents $(I_\mu)_-$ on transversals $(I_\mu)_+$ or $(I_\mu) \rightarrow (I_{-\mu})$. Further we have argued that the decomposition of tangents and transversals is linearly independent (η_1 is assumed of order 1 and locally injective). Thus t and v maps

$$(I)_+ \bigoplus (I)_- \rightarrow W_\Gamma \bigoplus W_L \rightarrow WF_a$$

23. MULTI-VALUEDNESS

We note that if γ is defined by an algebraic homomorphism h , then multi-valuedness for $h(\psi)$ corresponds to multi-valuedness for ψ , since the corresponding Puiseux-series is mapped on a Puiseux-series. Consider $\Sigma = \{P = 0\}$ for a polynomial P such that $d_\zeta P \neq 0$ on Σ and $F(\psi)(\zeta)d_\zeta P(\psi)(\zeta) = \sum_j b_j(\zeta)\psi^{j/\mu}$ where F is assumed holomorphic with respect to Σ and ψ is assumed reduced (assume a pseudo-base element). For $\phi = \psi^{1/\mu}$, we get $h(\phi)^\mu = h(\psi)$ and if h maps reduced (pseudo-base) elements on reduced (pseudo-base) elements, we get an expansion $\sum_j h(b_j\phi^j) = \sum_j h(b_j)h(\phi)^j$ with multi-valuedness of the same order as the given set. Consider

$$P_c(\psi) = \prod_{\zeta \in Eq} (c\psi - h(\psi))$$

where h is not assumed reduced. If h^2 is locally injective we consider the locally discrete set

$$Eq = \{\zeta \mid h(\psi)(\zeta) = h(\psi)(\zeta_0)\}$$

and the corresponding Puiseux-expansion $\sum b_j(\zeta)\phi^j$. This is mapped by h on another Puiseux-expansion and a corresponding discrete set

$$Eq' = \{\zeta \mid \psi(\zeta) = \psi(\zeta_0)\}$$

Let $\eta(\psi) = h(\psi)/\psi$.

Lemma 23.1. *Assume h such that $h^2 = 1$, then η does not change sign over Eq .*

The proof is trivial. Assume Eg is a discrete set and that the number of elements is 2. We then have $hP_c(\psi) = (ch(\psi) - \psi)^2 \geq 0$ (with $h^* = h$ and ψ real). Thus $P_c(\psi) \geq 0$ under these conditions. In the same manner, if the number of elements is odd ($= \mu - 1$) and ζ_0 such that $\eta(\psi) \geq c$, then we have $P_c(\psi) \leq 0$. The sign for $P_c(\psi)$ is dependent on ζ_0 and not on $\psi \in \text{nbhd}(I_c)$. Note that if $\eta(\psi)(\zeta) = c + P(\zeta)$ (in a domain of holomorphy) where P is a first degree polynomial. In this case $P_c(\psi) = P(\zeta_0)^{\mu-1} = (a_0 + a_1\zeta_0)^{\mu-1}$.

Consider now $\eta(\psi)(\zeta) = \mu + P_1(\zeta)$, with P_1 polynomial, then $h(\phi) = b_0 + b_1\phi + b_2\phi^2$. We can assume $b_0 = 0$ (h algebraic) and apply h again to get $\eta(\psi)(\zeta) = (1 - c_2(\zeta))/c_1(\zeta)$. We could consider $\{\phi, \phi^2\}$ as a pseudo-base for an ideal (I) and as before consider the distance to $V = N(h(I))$, $d(\zeta, V) \leq C |\zeta|^{c\rho} (c_1 |\phi|^{\frac{1}{\kappa+\sigma}} + c_2)^\rho$ with $\rho < 1$, where $\kappa + \sigma$ is a positive constant dependent on choice of neighborhood of the origin. Let $\Omega_\mu = \{\zeta \mid h(\psi)(\zeta) = \mu\psi(\zeta)\}$ for a constant μ . Assume $\psi = e^\varphi$ such that $\varphi \rightarrow -\infty$ as $\zeta \rightarrow 0$, over $H_h(\psi) = 0$ we further have $d_\zeta h(\psi) = \mu d_\zeta \psi$. For $\zeta \notin \Omega_\mu$ we have $h(\psi) = \sum_j c_j \psi^{j/\mu}$ and the image under t lies outside the tangent-line Γ . The corresponding trajectory determines a spiral approximation of Ω_μ . Starting with the Puiseux-expansion $h(\psi) = c_0 + c_1\psi^{\frac{1}{\mu}} + c_2\psi^{\frac{2}{\mu}}$ and assuming $h \sim_m g$ where g is an analytic homomorphism, we have $\tilde{t}h(\psi) \sim \sum_j c_j z_1^{\frac{j}{\mu}}$. Monotropy for the trajectory γ corresponds to translation of ζ that is $\gamma_g = \tau_\epsilon \gamma_h$, why $\tilde{t}g(\psi) = \sum [\tau_\epsilon c_j(\zeta)] z_1^{\frac{j}{\mu}}$ where \tilde{t} is the projection on the second variable of γ .

Remark:

The condition dw locally reduced for a homomorphism w deserves explanation. If ψ is singular for the dynamical system, it is through our conditions isolated. If $\{\psi_j\}$ is a sequence such that $d\psi_j \neq 0$ and $dw(\psi - \psi_j) \rightarrow 0$, then $\{\psi_j\}$ must be a regular approximation of ψ , that is $\psi - \psi_j \rightarrow 0$. In this sense dw is reduced over regular approximations.

24. HANGES' THEOREM

We can now generalize Hanges' theorem to pseudo-differential operators A_λ with foliation that is included in the zero-set to the polynomial corresponding (using t) to $X = \{\gamma \mid P(\gamma)(\zeta) = 0\}$. Assume γ an integral curve with a real branch in X and that the associated dynamical system has polynomial right hand sides, we then have $\tilde{\gamma}(U) \subset WF_a(u)$, for solutions to the homogeneous equation to A_λ and for U a neighborhood of the origin. Conversely, if γ is not a tangent to the origin and does not have a real branch in X , we know that $\tilde{\gamma}(U) \cap WF_a(u) = \emptyset$

Assume for a symbol a , holomorphic and of finite type and corresponding a self-adjoint operator, the intersection of the foliation and the transversal set is included in the zero-set to a holomorphic function F . Assume further that F is in a finitely generated ideal $(I) = \ker h$, where h is assumed (modulo monotropy) algebraic and such that h^2 is locally injective. We will here consider the case where $h(F)$ real. Further that $A \sim_\infty \tilde{F} + \Psi$ where Ψ is assumed regularizing and \tilde{F} is the operator corresponding to F . Hanges ([10]) studies the characteristic set

corresponding to the principal part of the symbol a . We propose that if F is defined by $h(F) = 0$, the homogeneous condition can be replaced by the following.

Assume t a real variable and consider $\zeta \rightarrow t\zeta$ such that $h(t^2F) \sim_m th(F)$. Define a continuous function s according to $s(t\zeta) = h(tF)(\zeta)$, then s is homogeneous of order $\frac{1}{2}$. Let further $r^2(t\zeta) = tr^2(\zeta)$, where r is analogous to s . A conical neighborhood of z_0 is mapped by a canonical homogeneous transformation on an open conical set in $(x, t, \xi, \tau) \in T^*(\mathbf{R}^{n+1}) \setminus 0$. We can now argue as Hanges ([10]) that is the operator corresponding to $r^2(\zeta)$ is reduced to canonical form $t \frac{\partial}{\partial t} - \tilde{B}(x, D_x)$ close to $(x_0, 0, \xi_0, 0)$ and $\tilde{B} \in L_c^0$ with proper support. We study a mapping $T: (I) \rightarrow \text{nbhd } \Gamma$, where Γ is the tangent line to the foliation and where the associated dynamical system is assumed to satisfy Bendixson's regularity conditions (cf. [1]). We consider $\Sigma = \{\zeta \mid \text{Tr}(\zeta) = 0\}$ and $\tilde{\Sigma} = \{\psi \mid h(\psi) = 0\}$. According to Bendixson (cf. [1] Ch.2, Theorem 9) if the dynamical system has polynomial part right hand sides of order 1, we have exactly four trajectories that run through the origin. Hanges considers correspondingly four half-bicharacteristics defined through $\gamma_j: J \rightarrow \Sigma_j$, $j = 1, 2$ $\gamma_j(0) = z_0$, where J is an open interval in \mathbf{R} containing 0. For $J_k = \{t \in J \mid (-1)^k t > 0\}$ $k = 1, 2$ we write $\gamma_{j,k} = \gamma_j \mid J_k$ for these. Note that in our case $\Sigma_j = \Sigma$. The following proposition is a direct consequence of ([10] Prop.1)

Proposition 24.1. *Assume $\gamma_{j,k}$ a characteristic such that $\gamma_{j,k}(0) = z_0$. If for a pseudo-differential operator A described above, $u \in \mathcal{D}'$, solves the homogeneous equation $Au = 0$ and $\gamma_{j,k} \cap WF_a(u) = \emptyset$, then $z_0 \notin WF_a(u)$.*

24.1. The non-homogeneous equation. Assume P a polynomial operator and $P(D)u = f$ in \mathcal{D}' for $u \in \mathcal{D}'$ and γ is a trajectory to the dynamical system such that $\tilde{\gamma} \cap WF_a(u) = \emptyset$

Lemma 24.2. *If $\gamma' \rightarrow P_0$ singular through regular points, we have that $\tilde{\gamma}' \cap WF_a(u) = \emptyset$*

Proof:

Since P_0 is mapped by t on the foliation to an operator of finite type, we have $\tilde{\gamma}' \rightarrow$ a finite value. The indicator of growth to an operator F is then the same as the indicator to $\tau_{\tilde{\gamma}'} F$ as $\gamma' \rightarrow P_0$ ($\varphi \rightarrow \pm\infty$). Note that we have not assumed that γ' is a trajectory to the dynamical system. \square

Note that if P^2 is hypoelliptic and self-adjoint, we have $WF(Pu) \subset WF(u)$ and if $u = Pv$ in \mathcal{D}' we have $WF(P^2v) = WF(v) \subset WF(Pv)$. The geometrical objects are considered as the same, it is only the names that are swapped. We assume that the latter case is used. We are going to prove that $WF(Pu) = \{\tilde{\gamma}\} \cup WF(u)$, where $\tilde{\gamma}$ is as in the lemma and P is a pseudo-differential operator with foliation in the zero-set to a polynomial F .

Lemma 24.3. *Assume γ a trajectory to the dynamical system and $\gamma \rightarrow P_0$ singular, with a tangent determined in P_0 , then there is a regular approximation of P_0 , for P as above.*

Proof:

The conditions give that all approximations γ of P_0 have a tangent determined (I_c) , c constant, in P_0 . As the foliation to the operator is finitely generated (in an

algebraic variety), we have the minimally defined case and there always is a regular approximation. \square

Remark:

Note that the conditions in the second lemma are satisfied if the dynamical system has polynomial right-hand sides.

Lemma 24.4. *Assume P_0 singular and mapped by t onto $N(I)$ for a finitely generated ideal (I) . If γ is a regular approximation of P_0 with tangent determined (I_c) constant, then γ is in the set of transversals.*

Assume now A a pseudo-differential operator of finite exponential type and with foliation in $N(F)$ for $F \in (I)$ polynomial and (I) a finitely generated ideal (the minimally defined situation). Assume $\tilde{\gamma} \cap WF_a(Au) = \emptyset$ and $\tilde{\gamma}$ transversal, that is γ is a regular approximation of P_0 , a singular point. If we have the representation $Au = \tilde{F}u + Ru$ where R is regularizing and \tilde{F} has zero's that include the foliation to A , then the generalized Hanges' result gives that $WF_a((A - \tilde{F})u) \cap \tilde{\gamma} = \emptyset$ and $WF_a(u) \cap \tilde{\gamma} = \emptyset$.

Proposition 24.5. *Assume A a pseudo-differential operator, self-adjoint and of finite exponential type. Further that the foliation is included in the zero-set to a polynomial or a difference of two polynomials in a finitely generated ideal. Assume every foliation point mapped on a singular point for the associated dynamical system, such that every trajectory that reaches such a singular point, has a tangent determined in this point. Then every regular approximation of the singular point is mapped in the set of transversals \tilde{S} , so that $WF_a(Au) = \tilde{S} \cup WF_a(u)$.*

25. FOLIATION IN A SEMI-ALGEBRAIC SET

Assume an ideal of holomorphy (I) defined by a polynomial P_μ and consider (\bar{I}) the Whitney-closure. Define the subspace $(I_\lambda) = \{\gamma \mid P_\mu(\gamma) \leq \lambda\}$ and the corresponding ideal (\tilde{I}_λ) under the mapping t . We will now discuss some properties of the class of operators that has foliation in intersection with transversal set included in the semi-algebraic set $\{\zeta \mid \tilde{P}_\mu \leq \lambda\}$.

If the symbol to our pseudo-differential operator has foliation in $\{\zeta \mid \tilde{P}_\mu \leq \lambda\}$ we can prove that the corresponding ideal (I) containing the polynomial P_μ , has a global pseudo-base. Assume (I) finitely generated such that the minimally defined zero-set case holds, that is assume over the semi-algebraic set that $P_\mu = 0 \implies dP_\mu \neq 0$. Singular points are now such that $\{P_\mu = \lambda\}$, λ constant and also the semi-algebraic set is minimally defined, since the singular points are the foliation to a polynomial P_μ , which is assumed to be with lineality (not reduced) and we have regular approximations over transversals. Note that for $\psi = e^\varphi$, as $(I_c) = (I_c)' = (I_c)''$ we have $P(\delta_j \gamma) = P(d\gamma)$ for all j , why in the minimally defined case, we must have for all j , $P_\mu \delta_j - \delta_j P_\mu \neq 0$.

Given exactness for $\eta (= h(\psi)/\psi)$ we can show existence of a global pseudo-base also for the ideal that defines the foliation (I) . Assume $P_\mu \in (I)$ and that the semi-algebraic set $\{P_\mu \leq \lambda\}$ is minimally defined such that we always have regular

approximation of singular points ($P_\mu(0) = \lambda$) and under these conditions every regular approximation has a tangent determined.

Proposition 25.1. *Consider for a pseudo-differential operator A , $Au = f$ in \mathcal{D}' , for $u \in \mathcal{D}'$, where we assume A has symbol with a foliation in a semi-algebraic set $\{P_\mu \leq \lambda\}$ that is minimally defined and where $P_\mu \in (I)$ is a polynomial and (I) is assumed to have a global pseudo-base. Then for \tilde{S} the transversal set, $WF_a(Au) = \tilde{S} \cup WF_a(u)$.*

Assume now $\gamma \in L_\mu$ and the foliation in a semi-algebraic set as before, this means that $\{\mu(d\gamma) \leq \lambda\}$ that is a bounded positively definite measure defined close to the foliation. Particularly if the ideal (I) is finitely generated and the index $\rho \leq 1/N$ for an integer N , we have for $M_\mu = \{\gamma \mid \mu(d\gamma) \leq \lambda\}$ that $|\mu(d\gamma)| \leq C |d\psi|^\rho$ for a positive constant C and conversely if $|d\psi| \leq \lambda^N$ we must have $\gamma \in M_\mu$.

Assume existence of a global pseudo-base for (I) and Φ_μ the generalized Cousin-integral (cf. [7]), then there are $\tilde{\Phi}_\mu$ of type 0. Assume $\tilde{\mu}$ such that $\tilde{\Phi}_\mu = \Phi_{\tilde{\mu}}$ (addition of measures with point-support), then the corresponding polynomial can be chosen reduced, that is with no foliation why the only singular point possible is the origin. The conclusion is that for every trajectory to the associated dynamical system that is tangent to M_μ holds that this trajectory will stay in a bounded set in phase space as $\varphi \rightarrow \pm\infty$. For M_μ the same conclusion holds μ -a.e.

26. A CASE OF INFINITELY GENERATED FOLIATION

We have earlier established that in the case where h^2 is locally injective and $\eta (= h(\psi)/\psi)$ exact, that we have a global pseudo-base for the ideal that generates the integral curves. Note that since h is reduced over the tangent space, if we form $(I) = \{\phi \mid \delta_j \phi \in (J) \mid j = 1, \dots, n\}$ we have a local pseudo-base for (I) and a global pseudobase for (J) . Assume now $\eta_j(\psi) = h^{j+1}(\psi)/h^j(\psi)$ without improvement (or degeneration) of behavior by iteration, that is we are assuming a not-finitely generated ideal. We give the following proposition, where τ denotes translation

Proposition 26.1. *Assume with $S_\tau \psi = \tau\psi/\psi$, $h(S_\tau \psi) \sim_m 0$ in the $|\zeta| \rightarrow -\infty$ and $\eta(\psi)$ not constant, then ψ is in a bounded set, symmetric with respect to the origin.*

Proof:

If η_j is constant then ψ must be on a transversal. Consider ψ_k in a regular approximation of ψ such that ψ_k are non-constant as well as $\eta_j(\psi_k)$. Assume further

$$(15) \quad h(S_{\tau_z} \psi) \sim_m 0 \quad |\zeta| \rightarrow \infty$$

particularly we have seen before that if h is algebraic, $S_{\tau_z} \psi \sim_m 0$ as $|\zeta| \rightarrow \infty$.

Obviously we have $\eta_j(\psi) \sim_m h(\psi)$, why if $|\eta_j(\psi) - h(\psi)| < 1/|\zeta|$ in the $|\zeta| \rightarrow -\infty$, we have $|\eta_j(\psi)| < 1/|\zeta| + |h(\psi)|$ and using the condition (15) we have that $\eta(\tau_z \psi/\psi) \sim_m 0$ as $|\zeta| \rightarrow \infty$ (modulo monotropy). Consider now

$\psi(z) = \tau_z \phi(x)/\phi(x)$ non-constant, why if ϕ is reduced, then ψ is bounded $\forall x, z$. According to (15) we have that $\psi\eta(\psi) \sim_m 0$ why we can claim that η is bounded in the $|\zeta|$ -infinity. Let (I) be the ϕ for which ψ is bounded when τ is compact. For reduced ϕ we know that the corresponding ϕ have a global pseudo-base. Obviously, the two ideals are locally equivalent in the $|\zeta|$ -infinity.

Let $h(\phi^2) - h^2(\phi) \sim_m 0$ such that $h(\phi^2 - h(\phi)) \sim_m 0$. Assume further that $\phi \neq 0$ in the $|\zeta|$ -infinity such that $\phi - \eta(\phi) \sim_m 0$. We then have that modulo monotropy $|\phi| < 1/|\zeta| + |\eta(\phi)|$. In this sense η is downward bounded in the infinity. Further symmetry with respect to the origin in ζ can be shown for regular approximations of (I_c) , c a constant. \square

Proposition 26.2. *Let (I) be the ideal of ϕ such that $\tau_z \phi/\phi \sim_m 0$ in the infinity and τ_z compact. Then (I) has a global pseudo-base.*

Proof:

Assume Schwartz-type topology and a compact translation. We can then assume the ideal (I) finitely generated. Further it is not difficult to prove, for instance using the quantitative version of Rückert's Nullstellensatz, that $\eta \sim_m 0$ over (I) . Thus the indicator for η over (I) is 0 and we can find a transform $\gamma \sim_m \eta$ over (I) , that is exact so that the ideal defined by γ has a global pseudo-base. \square

Assume the conditions in Proposition 26.1 further $d\eta \neq 0$ and continuous. Then $\psi \in B$ where B is a bounded set, symmetric with respect to the origin. Considered as a cone, $tB = \Gamma$ always has negative indicator, that is for ξ_0 finite and s a real number, we have $\langle sy, \xi_0 \rangle < 1 \Leftrightarrow \langle y, \xi_0 \rangle < 1/s \rightarrow 0$ as $s \rightarrow \infty$. We now know (cf. [21] Theorem 6.6) that in this case we have no contribution to the wave-front-set.

Proposition 26.3. *Assume ψ such that $h(S_\tau \psi) \sim_m 0$ in the infinity and η such that $d\eta(\psi) \neq 0$ and continuous, we then have that $\{t\psi\}$ is not mapped on to the wave-front-set.*

27. IMPLICATIONS ON MICROLOCAL ANALYSIS

Assume ζ_0 a point in the plane, such that $h(f)(\zeta_0) = 0$ of order N and consider a neighborhood W of ζ_0 , such that ζ_0 is an isolated zero. Then $(\zeta - \zeta_0)^N v(\zeta) = h(f)(\zeta)$, where v is real analytic and such that $v(\zeta_0) \neq 0$. Assume γ a real-analytic function such that $\gamma(x_0) = \zeta_0$ (implicit function theorem). Then, in x_0 , $f(\gamma(x_0)) = \text{constant}$. Let

$$X_c = \{x \mid f(\gamma(x)) = c\} \text{ for a constant } c.$$

If $f = \hat{u}$, for $u \in \mathcal{S}$ and if u has support on X_c , then $X_c \subset \text{sing supp } u$. Further, if we consider full lines in V_c , we can chose γ of order 1 and such that $h(f)(\gamma(x)) = c_N(x - x_0)^N v(\gamma(x))$. More precisely, in the Cousin integral approach, if $f = \tilde{\Phi} - \Phi$ and $e^\varphi = f$, we can give V_c as $\{\zeta \mid \varphi(\zeta) = \text{const.}\}$. Consider $V = \text{nbhd } \zeta_0$ and $f(\zeta_0) = c$. Let $\eta(V_c \cap V) \subset X_c \cap U$, for $U = \text{nbhd } x_0$, where γ, η are real-analytical functions (of order 1) and injective on these sets. Let $\zeta \in V_c$ and $u(\eta(\zeta)) = \hat{v}(\zeta)$, for $\hat{v} \in \mathcal{E}'(V)$. Assume \hat{v} is constant on $\Gamma \cap V$, for a

line Γ . Then, $\mathcal{F}^{-1}(u(\eta(\zeta))) = c\delta_\Gamma(x)$, for $\Gamma = \{\gamma(x)\}$ with $\delta_\Gamma(x) = \delta_0(x - \Gamma)$. Note that if $u \in \mathcal{S}'$, then $u(\eta(\zeta)) \in \mathcal{S}'(V)$.

The restriction (inclusion)-homomorphism r_Γ considered on \mathbf{R}^n $C_r^\infty \rightarrow C_r^\infty \rightarrow C^\infty$, can be extended to

$$\mathcal{E}' \rightarrow (C_r^\infty)'(U) \rightarrow (C_r^\infty)'(\Gamma),$$

where C_r^∞ denotes real-analytic functions. Assume β another homomorphism on \mathcal{E}' , such that $\beta(T)(x) = 0$ for $T \in \mathcal{E}'$ and $x \in \Gamma$, a line-segment, implies T is constant on this line-segment, that is $r'_\Gamma T = c\delta_\Gamma$. Let now β be defined with its support outside $\text{sing supp } T$. Consider for instance the class of distributions $\mathcal{E}'_\Omega(V)$ for open sets $\Omega \subset V$ as the class $\{T \in \mathcal{E}'(V) \mid \text{sing supp } T \subset \Omega\}$ and assume β with support on $V \setminus \Omega$. Let

$$W_h = \{(x, \gamma(x)) \in U \times V \mid \beta(T)(x) = 0\}$$

and

$$W_c = \{(\eta(\zeta), \zeta) \in U \times V \mid \zeta \in V_c\}.$$

Thus, over $\mathcal{E}'_\Omega(V)$ we have that β defines the foliation and $W_c = W_h$

Assume r'_L the restriction-homomorphism to a convex, compact set L and acting on \mathcal{E}' . If Γ is the polar to L , a line-segment through x , we have that also Γ is a line-segment and the functional $\mathcal{F}^{-1} r'_\Gamma \mathcal{F} T$ is portable by L . It also follows that r'_L is portable by L . We let $\mathcal{F} r'_L = r'_\Gamma \mathcal{F}$ in $\text{Exp}_{\rho, A}$. Assume γ, η real-analytic functions mapping L to Γ and back (inverse function theorem), we have the following proposition,

Proposition 27.1. *If $u \in \mathcal{E}'_\Omega(V)$, $x \in \text{sing supp } u$ and x on a line-segment L in Ω . Then there is a constant C , such that $r'_L u = C\delta_L$. If $f = \hat{u}$ and $\zeta \in V_c$ and on a line-segment Γ in V_c , then $r'_L u = C\delta_L$, for a constant C .*

Assume $\beta: C^\infty \rightarrow C^\infty$ and $\alpha = \beta\mathcal{F} - \mathcal{F}\beta: \mathcal{E}' \rightarrow C^\infty$ and for the restriction to a line-segment, $\mathcal{E}' \rightarrow C^\infty_\partial$. As $\beta u(x) - \beta u(-x) = \alpha\mathcal{F}u + \mathcal{F}\alpha u$, we have that $\beta u(x) - \beta u(-x) \in C^\infty(\text{nbhd } L)$. Assume $\beta u(x) - \beta u(-x) \neq 0$ on L and $\zeta \in U_c \setminus V_c$, $x = \eta(\zeta) \in L$. Then

$$\frac{1}{|x|^c} \leq C \mid \beta u(x) - \beta u(-x) \mid \leq C' (| \beta u(x) | + | \beta u(-x) |) \text{ on } L,$$

thus $x \notin \text{sing supp } u$ or $-x \notin \text{sing supp } u$. Assume M a set, symmetric around 0, $M = \text{supp } F$ for a real and real-analytic function F . Define

$$Eq^\pm = \{\zeta \in M \mid F(\zeta) = F(-\zeta)\}.$$

Then if $F_1 = \Pi_{\xi \in Eq}(\zeta - \xi(\zeta))F(\zeta)$ and $\text{supp } F_1 \subset M \setminus Eq^\pm$. In the same manner, if $Eq' = \{\zeta \in M \mid F_2(\zeta) = 0\}$, for F_2 real and real-analytic, we can define F'_1 with $\text{supp } F'_1 \subset M \setminus Eq'$. The conditions above are satisfied for $L \subset \text{supp } F'_1$.

Let $A = \{\zeta \mid h(\hat{u})(\zeta) = dh(\hat{u})(\zeta) = 0\}$. Note that \bar{J}_h as a closed ideal, can be considered as radical. Assume $h(f) = \alpha f$, for $\alpha \in H$ and h^2 injective. Then, $|f|^2 \in \bar{J}_h$ is equivalent with the proposition $h(|f|^2) = \text{constant}$. Further, if $h(|f|^2) = \beta |f|$, for a constant β , we conclude that $|f| = \text{constant}$.

Proposition 27.2. *Assume $N = \{\zeta \mid h(f)(\zeta) = 0\}$, $C_\Gamma = \{\zeta \mid dh(f)(\zeta) = 0\}$, $C = \cap_\Gamma C_\Gamma$ and $A = C \cap N$, then for Γ_1 a line through ζ , we have*

$$\begin{cases} |r_{\Gamma_1} f| = \text{constant for } \zeta \in C \\ r_{\Gamma_1} f = \text{constant for } \zeta \in A \end{cases}$$

28. SYMMETRY AND THE SINGULAR SUPPORT

Let $\Omega_A = \{\zeta \mid |\zeta| < A\}$ and for a small set U , $\Gamma_A^U = \{\zeta \in U \mid |\zeta| = A\}$. Let $\Omega_\epsilon = \{x \mid \frac{1}{2}\epsilon < |x| < \epsilon\}$ and $x = \eta(\zeta) = \sum_j a_j t^j$ the Puiseux-expansion corresponding to the singular points on Ω_A . Then for $\zeta \in \Omega_A$ and $\eta(\zeta) - \xi \in \Omega_\epsilon$, $|\eta(\zeta)| < |\xi| + \epsilon$. Assume as before, $f(\zeta) = \widehat{c}(\zeta) = u(\eta(\zeta))$ and further $\zeta \in \Omega_A$ implies $\eta(\zeta) \notin \text{sing supp } u$. In the construction of the Green-function, we use $T_\epsilon = \alpha_\epsilon g_\lambda$ for $\alpha_\epsilon \in \mathcal{D}$ with support in $|x| \leq \epsilon$ and with $\alpha_\epsilon(x) = 1$ for $|x| \leq \frac{1}{2}\epsilon$, it is a function only of $|x|$. g_λ is a tempered fundamental solution to $D_\lambda = \Delta - \lambda$. We then have $T_\epsilon(\eta(\zeta) - \xi) \in C^\infty(\Omega_\epsilon)$. Let $N_\lambda = N(D_\lambda)$. Then on $U_c \setminus N_\lambda$, $1 \leq |\widehat{D}_\lambda| |\widehat{g}_\lambda| \leq C |\xi|^{q-c} |\widehat{g}_\lambda|$, why if $c > q$, we can regard \widehat{g}_λ as reduced. If further f is reduced, for $\zeta \in U_c \setminus N_\lambda$, $|\eta(\zeta) - \xi|^\sigma \leq |\widehat{T}_\epsilon * c(\eta(\zeta))|$. If $\xi = \eta(v)$ for $v \in \text{nbhd } \zeta \subset \Omega_A$, $v = \zeta + \epsilon e^{i\phi}$ we have as η is real-analytical, for $|\zeta| < r < A$, that $|\eta(\zeta) - \xi| \leq \sum_\alpha C'_\alpha r^{-|\alpha|} \epsilon^{|\alpha|}$

We can regard $T_\epsilon \in \mathcal{E}'$ as very regular in \mathcal{D}'^F . Thus, T_ϵ is on the form $\delta_0 - \gamma_\epsilon$, with γ_ϵ regularizing and $\gamma_\epsilon(0) = 0$. Assume $\Gamma: \{\xi = \eta(\zeta)\}$. We have

$$\int_\Gamma T_\epsilon(\eta(\zeta) - \xi) u(\xi) d\xi = \int_\Gamma \delta_0(\eta(\zeta) - \xi) u(\xi) d\xi = \sum_{\zeta_j} u(\eta(\zeta_j)) = \sum_j^{\text{finite sum}} f(\zeta_j)$$

This is also the expression we expect, when we let $\epsilon \rightarrow 0$ in T_ϵ .

Assume $V = \text{nbhd } \zeta$ and $u(\eta(\zeta)) \in C^\infty(V)$. Let $U = \eta(V)$ such that $u(x) \in C^\infty(U)$ (also $\gamma(U) = V$). Let

$$W'_{L^2}(u) = \{\zeta \mid u(\eta(\zeta)) \in C^\infty(V') \mid \exists V' = \text{nbhd } \zeta \mid u \in L^2\}$$

Assume u has the property that

$$(16) \quad \eta(\zeta) \notin \text{sing supp}_{L^2} \text{Re } u \Rightarrow \eta(\zeta) \notin \text{sing supp}_{L^2} u$$

Then, obviously for u according to (16), we have $\zeta \in W'(\text{Re } u)$ implies $\zeta \in W'(\overline{u})$. We regard, for a polynomial $P(\zeta)$, $P\chi_{\Omega_A}$ as a bounded operator in L^2 . So, if for a constant β , $\beta\overline{P} = u(\eta)$ in L^2 , $\text{Re } P = \frac{1}{2}(P + P^*)$ and $\beta(P + P^*) = 2 \text{Re } u = u(\eta(\overline{\zeta})) + \overline{u}(\eta(\zeta))$

Lemma 28.1. *For u according to (16), we have $\zeta \in W'_{L^2}(\text{Re } u)$ implies $\overline{\zeta} \in W'_{L^2}(u)$*

In the general case, we have that the size of $V \subset W'_{L^2}$ is directly proportional to the distance from ζ to 0. Conversely to (16),

Proposition 28.2. *For $V \subset W'_{L^2}(u)$ and $V \subset W'_{L^2}(\overline{u})$, where V is symmetric with respect to the real axis, then $V \subset W'_{L^2}(\text{Re } u)$*

29. ON PRESERVATION OF FLATNESS

Let for a homomorphism β ,

$$I_C = \{g \in \overline{J}_h \mid \beta(f - g) = 0 \quad \exists f \in J_C^\infty\}$$

such that any $g \in I_C$ can be represented as $g = f + c$, for a constant c on Γ_A^U , where $f_\alpha(\eta) = 0$ for all α and $\eta \in C$. Let C be a cone with vertex in the origin and $f \in J_h^\infty(T^C)$, where $T^C = \mathbf{R}^n + iC$ and $J_h^\infty(f) = \{D^\alpha I_H(f)\}_\alpha$, then $I_H(f)(x + it\eta) - I_H(f)(x) = 0$, for $t\eta \in C$, why $C \subset \Delta_{\mathbf{C}}(I_H(f))$ and $I_H(f)$ is flat on T^C .

Assume $W = \{\zeta \mid D^\alpha I_H(f)(\zeta) = 0 \quad \forall \alpha\}$. If $D^\alpha I_H(f) = I_H(D^\alpha f)$ for all α , we have that h preserves flatness. Now consider

$$I_{B_\alpha}(f) = D_x^\alpha I_H(f) - I_H(D_y^\alpha f)$$

(also consider $I_{B'_\alpha}(f) = I_{(D_x^\alpha H - D_y^\alpha H)}(f)$, if ${}^t I_H = I_H^*$, we have $I_{B'_\alpha} = I_{B_\alpha}$).

Immediately, $I_{B_\alpha} = [D_x^\alpha I_H - \overline{D_x^\alpha I_H^*}] + [\overline{D_x^\alpha I_H^*} - (\overline{D_x^\alpha I_H^*})^*]$. Let $J(I_H) = I_H - I_H^*$. Then

$$I_{B_\alpha} = D_x^\alpha J(I_H) + J(D_x^\alpha I_H^*) + J(\overline{D_x^\alpha I_H^*})$$

If I_H is a bounded operator in L^2 , we have $J(I_H) = 2i \operatorname{Im} I_H$. If ${}^t I_H = I_H^*$ we have $J(I_H) = 0$ and if I_H is an integral operator with distribution kernel, then B_α can be neglected. With the condition that h^2 is locally injective, it is clear that h^2 preserves flatness and if $D^\alpha I_H(f) \in J_h$, $\forall \alpha$, we have that $cD^\alpha f = I_{[B_\alpha, H]}(f)$, for a constant c and since $f \in \overline{J}_h$, we see that h preserves flatness.

Proposition 29.1. *If for all α , $B_\alpha \prec\prec H$, we have in \mathcal{D}'^F , that h preserves flatness. Further, $\operatorname{Re} h^N \sim_\infty h^N$, for some N over V_c , so h^N preserves flatness. If also h^2 is locally injective, we have that h preserves flatness over \overline{J}_h .*

Proof:

Assume r^t the restriction homomorphism to $\Omega_t \supset V_c$, so that $\Omega_t \downarrow V_c$ as $t \uparrow \infty$. The proposition that $(rh)^t$ is reduced, is then a proposition that $(rh)^t = \text{const.}$ on Ω_t , that is if we let $2i \operatorname{Im} (rh) = 2(rh) - 2 \operatorname{Re} (rh) = (rh) - (rh)^*$, using geometric equivalence, $\operatorname{Re} (rh)^t \sim_\infty (rh)^t$. \square

Assume p and β homomorphisms, with $p = \frac{\delta}{\delta x_1} + \beta$, then $I_p = \ker p$ is an ideal. If $g \in I_p$ we see that

$$(17) \quad \left(\frac{\delta}{\delta x_1}\right)f[g - \beta g] + \left(\frac{\delta}{\delta x_1}\right)g[f - \beta f] = \left(\frac{\delta}{\delta x_1}\right)f\left(\frac{\delta}{\delta x_1}\right)g + \beta f \beta g$$

so $\left(\frac{\delta}{\delta x_1}\right)(fg) = \left(\frac{\delta}{\delta x_1}\right)f\left(\frac{\delta}{\delta x_1}\right)g - \beta(fg)$. If ζ is a zero of order ≥ 2 for g , (17) gives that

$$\beta g[\beta f + \left(\frac{\delta}{\delta x_1}\right)f] = 0$$

and if $\beta g \neq 0$ we have that $f \in I_p$. Let $V_2 = \{\zeta \mid g(\zeta) = \beta g(\zeta) = 0\}$, then $\left(\frac{\delta}{\delta x_1}\right)g[f - pf] = 0$, so if $\operatorname{ord}_\zeta g = 1$, we have $f = pf$ and thus $\left(\frac{\delta}{\delta x_1}\right)f = (1 - \beta)f$. Note that every geometric ideal $I(V_c)$ can be divided into $I^{(1)}(V_c) \cup I^{(2)}(V_c)$, where $I^{(2)}$ denotes $f \in I(V_c)$, such that $\operatorname{ord}_\zeta f \geq 2$, for $\zeta \in V_c$ and $I^{(1)}(V_c)$

denotes $f \in I(V_c)$ with simple zero's. Over $I^{(2)}$, $\frac{\delta}{\delta x}$ can be considered as a homomorphism, why it is sufficient to consider simple zero's.

30. ON THE TANGENTS TO THE WEYL GROUP, W

An analytic mapping between real vector spaces $f: E \rightarrow X$ can be factorized as

$$\begin{array}{ccc} E & \xrightarrow{\eta} & TS^n(E) \\ \downarrow & & \downarrow \tilde{f} \\ E^* & \xrightarrow{\hat{u}} & X \end{array}$$

so that $\tilde{f}(\eta(x)) = \hat{u}(\xi)$. Further, if for $U \subset X$, $\varphi: U \rightarrow E$, we have an induced linear mapping φ_* between the spaces of symmetric tensors and equivalence classes of symmetric tensors, with respect to change of coordinate maps, that are called point distributions. More precisely, given a chart (c, t) in a neighborhood of x , $\theta_c: TS^{(r)}(X) \rightarrow T_x^{(r)}(X)$ maps symmetric tensors on point distributions bijectively. There is uniquely, in $T_x^{(0)}(X)$ an element $\epsilon_x = \theta_c(1)$, independent on c , with center x . The field of point distributions D_t defined by t , with action by a Weyl group W , is given by $x \rightarrow t * \epsilon_x$. We have that

$$\tilde{f}(\theta_c(t)) = D_t f(x) = \langle t * \epsilon_x, f \rangle = \langle \epsilon_x, \check{t} * f \rangle = \langle t, f \circ \rho(x) \rangle$$

and if ρ traces Γ_A^U and if $D_t f = \text{const}$, we have $t = 1$, which implies $\theta_c(t) = \epsilon_x$ and if X is locally compact, we have $\epsilon_x = \delta_x$, the Dirac-measure. According to [3] (ch.III, Prop. 23), we can identify $V_W = \{f \mid \tau_g f = f \mid g \in W\}$ for $f = D_t f'$ with $f' \in C_r^\infty$ in a field of point distributions, with $U(W) = \{t \in \mathcal{T}^\infty(W) \mid \text{suppt } t \subset \{0\}\}$ where \mathcal{T}^∞ denotes distributions with finite support, thus $V_W \cong U(W)$. Further, $T_e(W)$ are the primitive elements in $U(W)$, that is $T_e(W) = \{f \otimes 1 + 1 \otimes f \mid f \in V_W\}$ and $T(W) = W \times T_e(W)$.

The field of point distributions defined by t , with action given by the Weyl-group W , (W can be generated by the lineality and for instance symmetry in a canonical way), is defined as $x \rightarrow t * \epsilon_x$, where with our conditions we can assume ϵ_x the Dirac-measure. If $D_{t'} h(f) = h_1(D_t f)$, then $\langle t', dh(f) \rangle = h_1(\langle t, df \rangle)$. Concerning the Weyl-group, if $g \in W_{h_1}$, we have $\tau_g h_1 = h_1$. Assume $\tau_g \pi = \pi \tau_g$. We then have $h\pi = \pi \tau_g h_1$. If $g' \in W_h$, we have $\tau_{g'} h\pi = h\pi = \pi \tau_g h_1$ implies $g = g'$. If $\tau_{g'} \pi = \pi \tau_g$ and $\tau_g h_1 = h_1$. Then $\pi \tau_g h_1 = \tau_{g'} \pi h_1 = \tau_{g'} h\pi = h\pi$, that is $\tau_{g'} h = h$ and $g' \sim g$. Essentially, existence of lineality and equivalent Weyl-groups, mean that the corresponding operators have the same micro-local properties.

Assume $D_{t'} h(f) = h_1 D_t(f)$, then $h_1(\langle t, df \rangle) = \langle t, h_1(df) \rangle$ and if $h_1 d = dh_1$, we have $0 = \langle t' - t, dh(f) \rangle + \langle t, d(h - h_1)(f) \rangle$. If $f \in \bar{J}_h$, we have $t' = t$. Assume $D_{\varphi_*(t)} = D_{t'} h = h_1 D_t$. Then $0 = \langle \varphi_*(t) - t', df \rangle + \langle t', d(f - h(f)) \rangle$,

thus if $f \sim h(f)$, we have $\varphi_*(t) \sim t'$. Let $D_{\varphi_*(t)}g = D_{t'}h$, for an algebraic homomorphism g , as in section 10. Then $f \in J_h^\epsilon$ means $\varphi_*(t) \sim t'$.

Consider the sequences

$$\begin{array}{ccccccc} (I) & D_t & X & \xrightarrow{p} & T(W) & \xrightarrow{\pi} & W \rightarrow 0 \\ & \downarrow h & & & \downarrow h_1 & & \downarrow \phi \\ (I') & D_{t'} & X' & \xrightarrow{p} & T(W') & \xrightarrow{\pi'} & W' \rightarrow 0 \end{array}$$

We note that $h_2 = T(\phi)$. If $\pi: W \times T_e(W) \rightarrow W$ is the canonical projection, $\pi^{-1}(e) = T_e(W)$. We then have $\pi' h_2 \pi^{-1} = \phi$ and if $\phi(e) = e (= 0)$, we have $h_2(T_e(W)) \subset T_e(W')$. If $\phi = id$, we have $h_2(T_e(W)) = T_e(W)$.

31. SOME REMARKS

Assume now the factorization

$$\begin{array}{ccc} X & & \xrightarrow{\eta} S(X) \\ \downarrow p & & \downarrow T_c \\ T(X) & \xrightarrow{\theta_\zeta} & TS(X) \end{array}$$

and $p = \pi^{-1}: X \rightarrow T(X)$, using the symmetric group S (cf. [2]). Thus $p(x) = \theta_c(T_c \eta(x)) = T_x(X)$. The proposition that η is constant, is then a proposition that $p(x) = 0$ or equivalently $T_c \eta(x) = 0$. If η is a submersion (for instance the orbital-mapping in the symmetric group), then using foliation, we have for all $x \in X$, existence of a subspace $V_x \subset X$ such that $T_x(V_x) \oplus T_x(X) = \ker {}_x T_c \eta$.

Assume $f \in J_h^\epsilon$ and if $D_{\phi_*(t)}g = D_{t'}h$ and if $h = g_1 g$, we have over J_h^ϵ , that g_1 is a translation. Monotropy as we have defined it, means particularly a factorization of an analytic homomorphism, by an algebraic homomorphism and a translation. Thus if $h: (I) \xrightarrow{g} (I') \xrightarrow{\tau_\epsilon} (I'')$ and $h_1: X \xrightarrow{g_1} X' \xrightarrow{\tau'_\epsilon} X''$. We have, $X' = \tau_\epsilon X''$ and $\langle t', dh(f) \rangle = \tau_\epsilon(\langle t', dg(f) \rangle)$. Note that the same t' can be used in the representation of monotropic functions.

Consider $h(\psi) = \tau_\epsilon w(\psi)$ such that

$$\begin{array}{ccc} w(\psi) & \longrightarrow & \alpha(\epsilon)w(\psi) \\ \downarrow & \nearrow & \\ h(\psi) & & \end{array}$$

Let $\Omega_\lambda = \{\epsilon \mid \frac{h(\psi)}{w(\psi)} = \alpha(\epsilon) \mid \alpha \in H \text{ not constant}\}$ and

$\Omega'_\lambda = \{\epsilon \mid \frac{\tau_\epsilon \psi}{\psi} \text{ not constant}\}$. When ψ reduced, Ω'_λ is bounded and symmetric with respect to the axes. If $w\tau_\epsilon = \tau'_\epsilon w$ where also $\tau_{\epsilon'} = \tau'_\epsilon$, we have $w(\frac{\tau_\epsilon \psi}{\psi}) = \frac{\tau_{\epsilon'} w(\psi)}{w(\psi)}$. As w preserves geometric convexity we see that Ω_λ is symmetric with respect to the axes. The same type of argument gives that it preserves boundedness. Note that Ω'_λ can be described as the set where α is not a polynomial outside the origin. As w maps polynomials on polynomials as does w^{-1} , the same description holds for Ω_λ . The same conclusion is for $\psi \in (J_A)$ that h and w have the same micro-local properties.

32. ALGEBRAICITY OF THE IDEAL

The mapping t . Given an ideal (J) we have if $f, g \in (J)$, then $\alpha f + \beta g \in (J)$ for constants α, β . But we do not necessarily have that $t(\alpha f + \beta g)$ is univalent. A necessary condition for geometric convexity in the plane must be local injectivity for t . Assuming this, if the ideal is defined by an algebraic homomorphism, we know that if $\{t\psi\}$ is a line, a closed curve or a spiral, then $\{th(\psi)\}$ is mapped on the same type of geometrical object. Since also h^{-1} is algebraic, the proposition can be reversed. We need however to assume $\{\psi\}$ reduced locally for univalentness, but to get the same geometric object, h need not be reduced. A necessary condition for an algebraic homomorphism to preserve geometric type of object in the plane is that the preimage is locally reduced and that t is locally injective, which is the case if we consider for instance regular approximations of singular ψ .

Assume $\psi \in (I)$ with a reduced pseudo-base F_1, \dots, F_ρ ordered after increasing order of zero and $h : F_j \rightarrow c_j F_j$ for constants c_j . Consider $Eq_1 = \{F_1 = cF_2 \quad \exists c\}$. As F_1 has a representation with locally isolated zero's, if $F_1 - cF_2 \neq 0$ on the boundary of a small disc Ω containing a zero to F_1 then according to Rouché's theorem, Eq_1 must be a locally discrete set. In the same manner for holomorphic coefficients, even if we have non-trivial foliation in the coefficients.

It is not difficult to prove that if (I') is finitely generated through an algebraic homomorphism, then $N(I')$ is algebraic. Let $\Omega' = N(\overline{I'})$ (closure in Whitney-norm) and $h : (I)(\Omega) \rightarrow (I')(\Omega')$. This must mean that Ω' is domain of holomorphy, if Ω is a domain of holomorphy. Since h maps constants on constants and polynomials on polynomials, we have that Ω' is a Stein-domain if Ω is a Stein-domain. Note that it is sufficient for the last result, to consider symbols on the form $1/R$, why it follows since h is locally bounded. In the case where (I') is infinitely generated, we can settle for studying regular approximations.

The measure μ . The disjoint decomposition of $\Gamma = \{\gamma(x)\}$ that we have found, can be used to define a submersion and corresponding transversal set. Define

$$J = \{g \quad dw(\mathcal{F}g) = d\mathcal{F}w(g)\}$$

and assume that dw is reduced, for a homomorphism w . We have

$$dw\mathcal{F} - \mathcal{F}dw = d\mathcal{F}w + B_1 + B_2$$

and $B_1 = (dw\mathcal{F} - d\mathcal{F}w) = 0$ on J , also $B_2 = (d\mathcal{F}w - \mathcal{F}dw) \in C^\infty$ for $g \in \mathcal{E}'$. We thus have $dw(\mathcal{F}g) = d\mathcal{F}w(g) + C^\infty$. For $g \in J$, we have $w(\mathcal{F}g) = \mathcal{F}w(g) + c$, for a constant c . Assume $\mathcal{F}w(g) = 0$ and

$$w(\mathcal{F}g)(\gamma(x)) = \widehat{u}(\xi) \text{ and } \Gamma = \{\xi = \gamma(x)\}$$

then for $g \in J$, we have $\widehat{u} = \text{const.}$ and $\mathcal{F}^{-1}(w\mathcal{F}g) = \delta_0$. Define

$$\Sigma' = \{x \quad \mathcal{F}^{-1}w(\mathcal{F}g) = w(g) \quad g \in J\}$$

Since $\widehat{H *' \varphi} \rightarrow \widehat{H}$ as $\widehat{\varphi} \rightarrow 1$ and analogously for $\widehat{H *'' \varphi}$, we see that $\Sigma = \Sigma'$. Thus, for $x \in \Sigma$, we have $w(g) = \delta_0$ and for $x \in \Delta_x^Q$, we have $w(g) = 0$. Further, if

$h(g) = w(\mathcal{F} g) - \mathcal{F} w(g)$, we have $h(g) = 0$ on Σ and if $g \in J$, we have $h(g) = \text{const.}$ on Δ_x^Q .

Assume $\mu = \mathcal{F} w - w\mathcal{F}$ a positively definite measure for an algebraic homomorphism w . We have the following proposition.

Proposition 32.1. *The measure μ is finitely generated for algebraic homomorphisms w and infinitely generated for analytic homomorphisms on the form $h = \tau_\epsilon w$.*

Proof:

According to Rückert's Nullstellensatz, if $\langle I_\Delta(\varphi), \psi \rangle = \langle \delta_x(\varphi), \psi \rangle$ for $\varphi, \psi \in H(V)$ where V is a bounded neighborhood of 0, then $\langle I_\Delta^N(\varphi), \psi \rangle = 0$ for large N . It is not difficult to prove that there is a $g_1 \in H$ such that $\mathcal{F} w(g^N) = \mathcal{F} [w, I_\Delta] (g_1^N) = \mathcal{F} w(I_\Delta^N(g_1)) = \text{constant}$. In the same manner there is a $g_0 \in H$ such that $w\mathcal{F} g^N = [w, I_\Delta] (\mathcal{F} g_0)^N = 0$. Note that if w is algebraic $[\hat{w}, I_\Delta] = [w, \hat{I}_\Delta] = [\hat{I}_\Delta, w] = [I_\Delta, \hat{w}]$. \square

In the case of $h = \tau_\epsilon w$, to prove that the measure μ is infinitely generated, it is sufficient to consider the one-variable case. For instance,

$\mu = [\hat{I}_\Delta, \tau_\epsilon w] - [\hat{I}_\Delta, e^{ix \cdot \epsilon} w]$. Let $T_1 = \frac{d}{dx} - ix$, $T_2 = \frac{d^2}{dx^2} + x^2$ and so on. Thus $\mu \sim \sum \frac{\epsilon^n}{n!} T_n w$ and we see that μ is infinitely generated for $\epsilon > 0$. It is clear that μ modulo monotropy is finitely generated. \square

Assume now that μ is algebraic in the tangent space, that is $d\mu^2(g) = d\mu(g^2)$ and consider $\mu_1 = \mathcal{F} d\mu - d\mu \mathcal{F}$. Thus μ_1 is finitely generated. The proof is completely analogous with the one given above and further it follows that if w is algebraic in the tangent space, then $d\mu$ is finitely generated. Let $\mu_{(2)}$ be the measure corresponding to w^2 , then $d\mu_{(2)}(g) \sim_m d\mu(g^2)$ for w algebraic. That is assume $g_1 * g_1$ a regular approximation of g^2 , such that $\hat{g}_1 \sim_m \hat{g}$. Then

$$d\mu_{(2)}(g) \sim_m d\hat{w}^2(g) - dw^2(\hat{g}_1) \rightarrow d\hat{w}^2(g) - dw(\hat{g}^2) = d(\hat{w}(g^2) - w(\hat{g}^2))$$

Given existence of g_1 as above we have that $\mu_{(\cdot)}$ is algebraic in the tangent space. Consider $h = \tau_\epsilon w$ with w algebraic. We have earlier considered $\Phi_\mu(\gamma) = \int_\gamma d\mu(\gamma)$ with $\gamma = (\psi, h(\psi))$ and ψ reduced. This is finitely generated and we have an estimate $|\Phi_\mu(\gamma)| \leq C |\psi|^\rho$ for the Puiseux-index ρ and a constant C .

Assume $\gamma = (\psi, h(\psi))$ with ψ reduced (or h reduced). If $\gamma \rightarrow P$, then P is on the curve and also \overline{P} is on the curve $\overline{\gamma}$. Assume that \overline{P} singular for γ . Then $d\gamma(\overline{P}) + d\overline{\gamma}(P) = 0$. This means that $\text{Re } \gamma$ is singular in P . Conversely, if P is singular for $\text{Re } \gamma$ and $d\gamma \rightarrow p$, then p must be purely imaginary. In the same manner if P is singular for $\text{Im } \gamma$ and $d\gamma \rightarrow p$, then p must be purely real. The conclusion is that if either $\text{Re } \gamma$ or $\text{Im } \gamma$ has P as singular point, then it can not be singular for the other unless it is singular for γ .

Denote $dh - hd = [dh] - [dh]^*$. If $h = h^*$ we have locally $dh - hd \sim 2i \text{Im } [dh]$. Sufficient conditions for a regular approximation are $h(dg) \neq 0$ ($\Rightarrow dg \neq 0$) and $dh(g) \neq 0$. Assume g satisfies these conditions and

$$(18) \quad dh(g) - h(dg) \sim \text{Im} [dh](g)$$

If $h \sim \bar{h}$, then locally $d[\text{Im } h] \sim \text{Im} [dh]$. Through the conditions for the dynamical system, if P is singular for $\text{Im} [dh]$ that is $((18) = 0)$, then P can not be singular for γ if it is not singular for $\text{Re } \gamma$. Thus if we assume real dominance, then singular points are given by the ones for $\text{Re } \gamma$. If $h^2(g) = h(g^2) + F$, where $dF \sim_m 0$, then there are dF_1 of type $-\infty$ such that $dF - dF_1$ is of type 0. We could say using the generalized Cousin integral representation, that $h^2(g) = h(g^2)$ in the tangent space modulo regularizing action.

32.1. Algebraicity in the tangent space. If h is not algebraic but algebraic in the tangent space, we have

$$dh^{-1}(g) = dh(1/g) = dh\left(\frac{1}{1-g'}\right) = d \sum_j h^j(g') = d \sum_j h(g'^j)$$

If h is algebraic in the tangent space, the corresponding η' is algebraic in the sense that $\eta'^{-1}(g) = gdh^{-1}(g) = (1-g')d \sum h^j(g')$. We can form the ideals

$$(J'_\lambda) = \{g \mid dh(g) \geq \lambda g\}$$

for $\lambda > 0$. We have that $dh^j(g')/g'$ does not change sign over (J'_λ) . If g' is real then $g'^2 \geq 0$ and also $g'd \sum h^j(g')$ does not change sign over (J'_λ) . For $\eta'^{-1}(g)$ we have constant sign on $(J'_\mu) \cap V_\pm$ where $V_\pm = \{g \mid \pm g > 0\}$. Thus $\eta'^{-1}(g)$ only changes sign on $\{g = 0\}$.

33. ANALYTIC SETS AND SYMMETRY

33.1. A weighted lineality. Assume $h(f) = P/Q$ for polynomials P, Q , where Q is assumed hypoelliptic and self-adjoint. Let $I_{G_{\tau_\eta}} = \tau_\eta I_H \tau_\eta$. Thus

$$\frac{Q(x+\eta)}{Q(x)} I_{G_{\tau_\eta}} = \left(1 + \sum_{\alpha \neq 0} \frac{Q_\alpha(x)}{Q(x)} \eta^\alpha\right) I_{G_{\tau_\eta}}$$

for $|\eta| < R$, η real and R finite and where $Q_\alpha/Q \rightarrow 0$ as $|x| \rightarrow \infty$. Note that the condition Q hypoelliptic, implies that there are $|x|$ large such that $Q(x+\eta) \neq Q(x)$ for $\eta \neq 0$. We now have

$$I_{G_{\tau_\eta}}(f) - P/Q = \left(\sum_{\alpha \neq 0} \frac{Q_\alpha(x)}{Q(x)} \eta^\alpha\right) I_{G_{\tau_\eta}}(f)$$

We now introduce a "weighted" lineality $\Delta_x^Q = \{\eta \mid \frac{Q(x+\eta)}{Q(x)} I_{G_{\tau_\eta}} = I_H \quad \forall x\}$ where the term $\frac{R}{Q}h(f)$ is assumed insignificant. Let $\tau_\eta I_H \tau_\eta = \widehat{g}_\eta$, so that

$$\Delta_x^Q = \{\eta \mid \sum'_\alpha (q_\alpha(x) \eta^\alpha) \widehat{g}_\eta(f) = P/Q\} = \{\eta \mid \sum'_\alpha q_\alpha(x) (\widehat{D^\alpha g_\eta})(f) = P/Q\},$$

where Σ' denotes a finite sum, since we have assumed Q a polynomial. Further, we assume $q_\alpha(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $|\eta| < R$. Let

$$\widehat{\Delta}_x^Q = \{\eta \mid \sum'_\alpha q_\alpha(x) (D_\eta^\alpha g_\eta)(f) = (P/Q) \delta_0(\eta)\}. \text{ Thus,}$$

$I_{G_{\tau_\eta}} - P/Q = \sum'_\alpha q_\alpha(x) (\widehat{D_\eta^\alpha g_\eta})(f)$. Note that $\{\eta \mid \sum' = 0 \quad \forall x\}$ corresponds to the set $\Delta_{x,y}(h)$ in the Weyl-calculus, that is invariance for translation with η in both variables separately for the kernel H . Further if η is considered on a

line-segment L , then $\tilde{\Delta}_x^Q$ is Δ_x^Q considered on the dual line-segment Γ , \mathcal{F}
 $r_L = r_\Gamma \mathcal{F}$ in $Exp_{\rho,A}$.

33.2. Rückert for monotropic homomorphisms. Assume $h(f) = \tau_\epsilon g(f)$ as earlier and $h(f)$ with a zero in a point $a \in \mathbf{C}^n$, which is not of infinite order. We assume $g(f) \neq 0$ in a and in a neighborhood of a . Consider now translation along a transversal emanating from the point a . We have seen that $\tau_\epsilon^N g(f) = 0$ is modulo monotropy equivalent with $\tau_\epsilon g^N(f) = 0$ and further $h^N(f) = 0$. (Note that ϵ depends on N, f and a). Assume $N(J_g)$ the zero-set to the ideal corresponding to g , is an analytic set. We assume that it has positive dimension locally (that is near the translated a). Then there is an irreducible, given a stratification of $N(J_g)$, of dimension > 0 . This gives a neighborhood of the point a , where $h^N(f) = 0$

Lemma 33.1. *If h is an analytic homomorphism with a zero in a point a , with factorization $h(f) = \tau_\epsilon g(f)$, where g is an algebraic homomorphism with $N(J_g)$ locally of positive dimension, then h^N has an infinite zero in a , for $N \geq N_0$, with N_0 positive.*

33.3. The set of symmetry, $\Sigma_{x,y}$. Let

$\Sigma_{x,y} = \{\eta \mid H(x+\eta, y-\eta) = H(x, y) \quad \forall x, y \in \mathbf{C}^n\}$ for the kernel to a continuous linear operator h on $L^2 \cap H$. Assume to begin with that the kernel $H(x, y)$ is in $L^2 \cap H$, such that h is compact. Then $H(x+\eta, y) = H(x, y+\eta) \Leftrightarrow \eta \in \Sigma_{x,y}$ for all x, y . Particularly, $\Delta_{x,y} \subset \Sigma_{x,y}$ and with the conditions on $H(x, y)$ above, we have that h has slow oscillation over $\Sigma_{x,y}$. If $\Sigma_{x,y}^*$ is the set corresponding the adjoint operator h^* , we have for $h^* = h$ that $\Sigma_{x,y}^* = \Sigma_{x,y}$. Assume

$$\Sigma_{x,y}^{As} = \{\eta \mid H(x+\eta, y) = H(x, y+\eta) \quad |x| \rightarrow \infty, \quad |y| \rightarrow \infty\}$$

and correspondingly for $\Delta_{x,As}^Q$, we then have for η such that $|\eta| < R$, R finite, that $\Sigma_{x,y}^{As} = \Delta_{x,As}^Q$, if $I_{G_{\tau_\eta}} = \tau_\eta I_H \tau_\eta$ is bounded. Assume

$$\tilde{\Sigma}_{x,y}^{As} = \{\eta \mid H(x+\eta, y) = H(x, y+\eta) \quad x, y \in V\}$$

where V is a set of positive dimension containing the infinity. If $\eta \in \Sigma_{x,y}$ and $h^* = h$, we must have $\eta \notin \Delta_x^Q$ and conversely if $\eta \in \Delta_{x,As}^Q \setminus \Delta_x^Q$ then $\eta \in \Sigma_{x,y}^{As}$.

Lemma 33.2. *If $\eta \in \tilde{\Sigma}_{x,y}^{As}$ and $h^* = h$ over $L^2 \cap H$, then $\eta \in \Sigma_{x,y}$*

Proof:

Assume h is such that $I_{G_{\tau_\eta}} = I_H$ on a set of positive dimension in the infinity, for η finite, then the proposition follows from Lemma 33.1 \square .

Assume $h_1(x, y, \eta) = H(x+\eta, y) - H(x, y+\eta)$. If h_1 is entire in (x, y) , we must have for $\eta \in \Sigma_{x,y}^{As}$, that h_1 is identically 0. Otherwise, we assume h_1 holomorphic in a domain Ω containing the infinity. If h_1 is 0 on a set $V \subset \Omega$ of positive complex dimension (we assume V contains the infinity), then $h_1 = 0$ on Ω . The definition of H can be extended outside Ω to a constant, or for H developed in a pseudo-base, with constant coefficients, to an entire function. Again, if $\eta \in \Sigma_{x,y}^{As}$, where h is according to the conditions in Lemma 33.1, then there is a N

sufficiently large, such that $(I_{G_{\tau_\eta}} - I_H)^N = 0$ on a set of positive dimension close to the infinity. Taking monotropy in to account, this means that $I_{G_{\tau_\eta}}^N \sim_m I_H^N$.

The conclusion is thus, that for $h = h^*$ bounded over $L^2 \cap H$, if we consider the sets corresponding to the iterated operators h^N ,

$$(19) \quad \Delta_{x,As}^Q \sim_m \Delta_x^Q \cup \Sigma_{x,y} \text{ a disjoint union}$$

Note that given the division $\eta \in \Delta_{As}^Q$, $\eta = \eta_1 + \eta_2$ with $\eta_2 \in \Sigma_{x,y}^{As}$ and if we assume $g_\eta = \tau_\eta F = (\tau_\eta I_{G'} \tau_\eta) \tau_{\eta_2} \tau_{\eta_1} F$, where F is a self-adjoint integral operator with kernel representation such that $F \sim h^N$, we have the representation $g_\eta = \text{Re}(\tau_{\eta_1} F)$ for $\eta \in \Delta_{x,As}^Q$.

In the case where h does not have a kernel in $L^2 \cap H$, we still assume $\eta \in \Delta_{x,As}^Q$. This means particularly, that given that the translation is bounded, for $|\eta|$ finite, h and h^* have slow oscillation relative Q . Assume instead that h^{-1} is bounded, then obviously $\Sigma_{x,y}^{As} \subset \Delta_{x,As}^Q$. If we assume h^N locally injective, for large N , we also have the opposite inclusion, $\Delta_{x,As}^Q \subset \Sigma_{x,y}^{As}$. Further, we know that under these conditions, if $\eta \in \Delta_{x,As}^Q$, then for t real, $t\eta \in \Delta_{x,As}^Q$ as $t \rightarrow \infty$ since the translation is a compact operator over reduced elements. Naturally, the decomposition (19) holds for the iterated operators also without the condition that the operator is bounded on $L^2 \cap H$.

Assume now h is not self-adjoint, but such that $h \sim_\infty \text{Re } h$. Consider $I_{G_{\tau_\eta}}$ as a function of a complex η and let G_R be the kernel corresponding to $\tau_\eta \text{Re } I_H \tau_\eta = I_{G_R}$ and in the same way, let H_R be the kernel corresponding to $\text{Re } I_H$. Both operators are considered as locally defined in z, w . Thus, $G_R(z, w; \eta) = \frac{1}{2}(H(z + \eta, w - \bar{\eta}) + \overline{H}(z + \eta, w - \eta))$. Over $t\eta \in \Sigma(\text{Re } h) = \Sigma^*(\text{Re } h)$, for all real t , we have

$$G_R(z - \eta, w + \eta; \eta) = H_R(z, w) \Leftrightarrow \frac{1}{2}(H(z, w + 2i \text{Im } \eta) + \overline{H}(z, w)) = \frac{1}{2}(H(z, w) + \overline{H}(z, w))$$

The necessary condition is thus that $2i \text{Im } \eta \in \Delta({}^t I_H)$. Conversely, if $2i \text{Im } \eta \in \Delta_{z,w}(I_H)$, where I_H is locally defined in z, w , then $t\eta \in \Sigma_{z,w}(\text{Re } h)$, for all real t .

33.4. Behavior at the infinity. Note that for $\eta \in \Sigma_{x,y}^{As}$, we have that a constant value is preserved for h_1 in (x, y) , in the sense of Cousin [6], why it is natural to consider the quotient-topology $I_{h_1}(f) = P/Q$, and $f = P'/R$.

Lemma 33.3. *If P is a polynomial with a real zero in the infinity, then the set of complex zero's to P close to the infinity does not have positive complex dimension.*

This means that P does not have an infinite zero in the complex sense. If h is such that h^2 is injective (assume f reduced), the proposition that the zero in the infinity has complex dimension is that it is a regular point in the zero-set to $h(f) = P/Q$. Assume $f = P'/R$ an entire function, then there exists h , a

differential operator or $h \sim_m$ differential operator, such that $f \in \overline{J}_h$. Assume further that $R^* = R$ and hypoelliptic. The local condition

$$(20) \quad Q(D)(P'/R) = \text{constant}$$

means that, for an appropriate test-function φ ,
 $\sum_{\alpha} Q_{\alpha} \varphi D^{\alpha}(P'/R) = \sum_{\alpha, \beta} Q_{\alpha} \varphi \frac{P'^{(\beta_1)} R^{(\beta_2)} \dots R^{(\beta_s)}}{R^{(s)}} = \text{constant}$. Assume $P'^{(\beta_1)}/R \rightarrow 0$ as $|\xi| \rightarrow \infty$ for all $\beta_1 \neq 0$ (slow oscillation relative R). A necessary condition for the condition (20) is that P' and Q are in some sense equivalent. We will settle for $P' \sim_m R$, that is $f \in J_h^{\epsilon}$ and where $h(f) = \text{constant}$ we have $|1 - g(P'/R)| < \epsilon$, for an algebraic homomorphism g . Particularly, if $g = 1$, $P' \sim_m R$. We can now study the ideals of polynomials

$$(J_Q) = \{P \quad \exists R \text{ hypoelliptic and self-adjoint } P \prec\prec R \quad P \sim_m R\}$$

and

$$(J'_Q) = \{P \quad \exists R \text{ hypoelliptic and self-adjoint } \delta P \prec\prec R \quad P \sim_m R\}$$

33.5. Some results for (J'_Q) . Assume f symmetric and Df corresponding to a regularizing operator, for instance $Df \prec\prec I$, with $f = P/Q$, where $D = D_{x_1} \dots D_{x_n}$.

Lemma 33.4. *Assume f as above and with Q hypoelliptic and self-adjoint. Then $D^{\beta_j} P \prec\prec Q$ for all $\beta_j \neq 0 \Leftrightarrow \text{Im } P \prec\prec Q$*

Proof:

It is not difficult to prove that for $C_{\varphi, f} = f\varphi - \varphi f$, where φ is a real test-function, arbitrary in C_0^{∞} ,

$$(21) \quad C_{\varphi, f} \prec\prec I \quad \forall \varphi \in C_0^{\infty} \Leftrightarrow D^{\beta_j} f \prec\prec I \quad \forall \beta_j \neq 0$$

Particularly, we have that,

Lemma 33.5. *If f has finite type and $f = R/S$, where $S = S^*$ and hypoelliptic, if $Df \prec\prec I$, then $\delta R \prec\prec S$.*

If Q is real, we obviously have $\text{Im } f \sim_{\infty} (\text{Im } P)/Q$. If Q is only self-adjoint, we have if we assume $P \prec Q$ and Q hypoelliptic, we get the same conclusion. With these conditions on Q further, $D(\frac{P}{Q}) \sim_{\infty} \frac{DP}{Q}$ and through the conditions on f , $f \sim_{\infty} \overline{f}$, so if we assume $(D^{\beta_j} P)/Q \sim_{\infty} D^{\beta_j}(\frac{P}{Q}) \prec\prec I$, for all $\beta_j \neq 0$, then according to (21) $C_{\varphi, f} \prec\prec I$ with φ arbitrary in C_0^{∞} , why

$\text{Im } f \sim_{\infty} (\text{Im } P)/Q \prec\prec I$. The converse follows in the same manner. Thus, $\text{Im } f \prec\prec I$ implies $C_{\varphi, f} \prec\prec I$ and according to (21), $D^{\beta_j} f \prec\prec I$ for all $\beta_j \neq 0$ and we are done. \square

Note that if for P, Q polynomials, $P - Q$ has a real zero in the infinity and if $(P - Q)^N$ has a zero of complex dimension in the infinity, then $(P - Q)^N$ can not be a polynomial. However, $(P - Q)^N$ has a ps.d.o realization with a regularizing action and with a polynomial part. Note that if P_j is reduced for all j , we only have to deal with isolated zero's, but $\sum_j P_j$ may well have a zero of infinite order. The condition, $\sum_j P_j \in \text{rad}(I_{RED})$ implies that we do not have a zero in the infinity, but $(\sum_j P_j)^N \sim_m \sum_j P_j^{\mu_j}$, for iteration indexes μ_j , where the right side may have a zero of infinite order. Assume V a set of positive complex dimension

containing the infinity and $h(f) = \sum_j P_j$, such that h satisfies the conditions in Lemma 5-4, then $\sum_j h^N(P_j)$ has an infinite zero in the infinity and we can assume $h^N(f) \in J_V^\infty$. Note that J_V^∞ denotes the ideal of symbols with an infinite zero on a set of positive complex dimension V . This corresponds to an operator with regularizing action in a ps.d.o realization.

33.6. Behavior in the origin. Assume $g = R/S$, with S hypoelliptic and self-adjoint and $Dg \rightarrow 0$ as $|x| \rightarrow \infty$. Let $J_\infty = \{g \mid D(f - g) = 0 \quad \exists f \in J_V^\infty\}$, that is, $g = f + c$ for a constant c in the infinity. The conditions on g mean particularly that $g \in (I)'_{\mathcal{Q}}$. Note that J_∞ is locally defined and that V is assumed to have positive complex dimension and that V contains the infinity. Particularly, if $g \rightarrow 0$ as $|x| \rightarrow \infty$, we have that $g \in J_\infty$ can be approximated by J_V^∞ .

Assuming $zP(z) = F(\frac{1}{z})$, for a polynomial P and F holomorphic for $0 < |z| < A < \infty$. If F has an infinite zero in the infinity, we can not draw a conclusion concerning the order of zero for P in the origin, since this corresponds to an essential singularity in the general case, as $z \rightarrow 0$. However, since $zP(z) \rightarrow 0$ as $z \rightarrow 0$, for all polynomials P , we could at least formally say that if $P \in J_\infty$, $zP(z)$ has a zero of infinite order in the origin.

Assume V a complex domain that contains the infinity and $u \in C^0(V)$. We know that $\log |u|$ is a sub harmonic function on $V \setminus Z_u$ and that $\log |u| \equiv -\infty$ on Z_u . Let $u = e^g$, then Z_u becomes the polar set to $-g$ (u is also holomorphic on V). Let $(I) = \{g \mid e^g \rightarrow 0 \quad |x| \rightarrow \infty\}$. For a complex-valued g , the corresponding ideal is not closed, but we can assume $g \sim_\infty \operatorname{Re} g$ or consider $\operatorname{rad}(I)$, to get a closed ideal.

Assume $\tilde{h}(e^\varphi) = e^{h(\varphi)}$ with $\tilde{h}(f) = \tau_\epsilon \tilde{g}(f)$ and $f = e^\varphi$. Then we have that $N(\tilde{h}(f))$ is the polar set to $h(\varphi)$. A comparison between the dimensions for $N(\tilde{h}(f))$ and $N(\tilde{g}(f))$ corresponds to a comparison between the dimensions for the polar sets to $h(\varphi)$ and $g(\varphi)$ respectively. Note that we may have $\dim V = 1$ for $V \sim_m V'$ and $\dim V' = 0$, this because ϵ depends on the choice of point x . Further, note that $D_t \tilde{h}(f) = \tau_\epsilon D_t \tilde{g}(f) = 0$, means $\tilde{h}_*(t) = \tilde{g}_*(t) = 0$ or $dh(\varphi) = 0$. Further, $\tilde{g}(f) = \tilde{g} \tilde{h}^{-1}(e^{h(\varphi)})$, so the latter condition is $dh(\varphi) = \tau_\epsilon d(h^{-1}gh(\varphi)) = 0$. If \tilde{h} is self-adjoint, we have $d\tilde{h}(f) = 0$ iff $f = 0$ without reducedness for h . Further, $\dim N(f) = \dim A$, where $A = \operatorname{sng} N(\tilde{h}(f))$.

33.7. Schwartz-type topology on the phase-ideal. We will in this section refer to [21] for notation and results. A contour Γ , is a C^∞ -mapping $W \rightarrow \mathbf{R}^n$, where $W \subset \mathbf{R}^n$ is open and we have Γ 1-1 and also $d\Gamma$ 1-1. A contour is “good” for a function φ , if $\tau_\eta \varphi - \varphi \leq -C |\eta|^2$ for a positive constant C and for η traces Γ . Here φ is assumed continuous $\Omega \rightarrow \mathbf{R}$, with $\Omega \subset \mathbf{C}^k$. For $u \in H_{\varphi, x}^{loc}$, we define $I_\Gamma(\lambda) = e^{-\lambda \varphi(0)} \int_\Gamma e^{\lambda \varphi(y)} u_\lambda(y) dy$. Over such φ , we claim that τ_η is a compact operator. Note that φ can be regarded as the monotropic function to a corresponding holomorphic function, further that the germs are locally finite-dimensional. Further, if $f \sim_m g$ and $g \rightarrow 0$, along complex lines, then

$|f - g| \leq Q_\epsilon(\frac{1}{t})$, for $Q(0) = \epsilon$, as $t \rightarrow \infty$. Particularly, we have $f \sim g$ in H_φ^{loc} , that is the difference gives an exponentially small contribution to the integral I_Γ .

The condition that $\eta \in \Delta_{x,As}^Q$ implies that the translation is weakly compact over h and $\varphi \in (I)'_Q$, that is $\frac{Q(x+\eta)}{Q(x)}\tau_\eta h(\tau_\eta \varphi) \rightarrow h(\varphi)$ as $|x| \rightarrow \infty$ and symmetrically as $|y| \rightarrow \infty$. Thus $\tau_\eta \varphi$ has to converge as $|x| \rightarrow \infty$ and τ_η is weakly closed over φ . By realizing τ_η as a measure over $(I)'_Q$, τ_η can be considered as weakly compact. Particularly, the extension used to define the Schwartz-type topology, is weakly compact.

Note that, if $T_\eta(h) = \tau_\eta h \tau_\eta$, for $\eta \in \Delta_{x,As}^Q$, we have that T_η is a compact operator, even though τ_η is not compact unless h is reduced. We define $\Delta_x^Q = \{\eta \mid (T_\eta - I)(Qh)(x) = 0 \quad \forall x\}$, where T_η is assumed compact over (Qh) . We have seen that modulo monotropy, $\Delta_x^Q = 0$, for the set corresponding to h^N , for all $N \geq N_0$, for some N_0 . Thus, Δ_x^Q has, for all N , representation with locally isolated points and can be realized by a locally injective, also C^∞ mapping. As $\Delta_x^Q \cap d\Delta_x^Q \subset \Delta_x^Q$, as an analytic set, the same argument can be applied on this set. For an operator on L^2 , on the form $A = T_\eta - I$, under the assumption that $\eta \notin \Delta_x^Q$, we have $N(A_\eta^N) = N(A_\eta^{N+1})$, $N > N_0$, for some N_0 . Modulo monotropy, we have $\varphi \in N(A_\eta^N)$ implies $\eta \in \Delta_{x,(N)}^Q$ and through the conditions, we see that $\eta = 0$.

Assume h localized to $2 \operatorname{Re} h = h(x + \bar{\eta}) + \bar{h}(x + \eta)$ as earlier. We then know that $\Delta_x^Q(\operatorname{Re} h) = \{0\}$. Thus, $\eta \in \Delta_x^Q(h^*) \Leftrightarrow \bar{\eta} \in \Delta_x^Q(h)$. If we have $h^* = h$, then Δ_x^Q is symmetric with respect to the real axis. Further, $\tau_\eta Qh = Qh\tau_{-\eta}$, why Δ_x^Q is symmetric with respect to the origin. Assume $\Delta_x^Q \subset U_c$, where U_c is a convex neighborhood of the origin. This neighborhood can be seen as a porteur for S_{τ_η} , and $\widehat{S_{\tau_\eta}}$ has indicator $-\infty$ over the porteur and (I_1) .

33.8. A generalization of the Cousin integral. The problem is to determine if I_Γ can be used to generalize the Cousin integral. Assume $\varphi(x) = \log(x - y)$. If Γ describes a circle-segment, then $\Phi = \int_\Gamma (\frac{\delta}{\delta x} e^{\lambda \varphi}) u_\lambda(x, z) dx$ is on the form of a Cousin integral. Φ is a representation of $F_\lambda = (x - y)^\lambda u_\lambda$ and if $u_\lambda \neq 0$ in a point y , then F_λ has a zero of order λ in this point. Further, Φ has analytic continuation along a transversal to the circle-segment Γ . Note that if Γ has a disjoint decomposition as an analytic set, for instance using properties for the phase-function, then the representative Φ has a corresponding decomposition. Assume the phase defined, modulo monotropy, through a homomorphism, such that $\Gamma = \Delta_x^Q \cup \Sigma_{x,y}$ disjointly and $\varphi^k \sim_m \tau_{x'} \operatorname{Re} \tau_{x'} \phi$, for an analytic ϕ and $x' \in \Delta_x^Q$, then

$$\Phi = \int_\Gamma (\frac{\delta}{\delta x} e^{\lambda \varphi^k(x)}) u_\lambda(x, z) dx = \int_{\Sigma_{x,y}} \int_{\Delta_x^Q} (\frac{\delta}{\delta x} e^{\lambda \tau_{x''} \operatorname{Re} \phi(x')}) u_\lambda(x, z) dx$$

Note that the integral $\int_{\Gamma} e^{\lambda h^k(\psi)} u_{\lambda} dx$, corresponds to $I_{\Gamma} I_H^k(u_{\lambda}(x, z))$, where $h^k = \operatorname{Re} h^k$.

Assume $\varphi = h(\psi)$, where $\varphi \in (I)_{\mathcal{Q}}'$ and where h is assumed self-adjoint in L^2 and self-transposed in the duality between H' and H . We then have $\varphi^k \sim_m h^k(\psi)$. Let \tilde{h} be defined by $e^{\lambda h^k(\psi)} = \tilde{h}^k(e^{\lambda \psi})$. By considering $I_{\Gamma}(u_{\lambda})(y)$ as functional $H_x \rightarrow H'_y$ and using Stoke's formula, if \tilde{h} is defined by $\tilde{d}\tilde{h} = \tilde{h}d$, we see that

$$\int_{\Gamma} e^{\lambda h^k(\psi)(x)} u_{\lambda}(x, z) dx = \int_{\Gamma} \tilde{h}^k(e^{\lambda \psi(x)}) u_{\lambda}(x, z) dx = \int_{\Gamma} e^{\lambda \psi(x)} \widetilde{\tilde{h}^k}(u_{\lambda})(x, z) dx$$

If k is selected so that $\widetilde{\tilde{h}^k}$ is linear, over the "eigen-spaces" to this operator, ψ can be selected as the real part of a holomorphic function. Also note that we can use that $\tilde{d}\tilde{h}$ is reduced, that is for u_{λ} in the symbol-space, there is a local constant c such that

$$\int_{\Gamma} e^{\lambda \psi} \widetilde{\tilde{h}^k}(u_{\lambda}) dx = c \int_{\Gamma} e^{\lambda \psi} \tilde{h}^k(u_{\lambda}) dx$$

Let $\Delta_{x,(2)}^Q$ be the weighted lineality corresponding to h^2 . We now claim that $\Delta_{x,(2)}^Q \subset \Delta_{x,(1)}^Q$, but the converse does not necessarily hold. Assume $h^* = h$ and let $\tau_{\eta} h = h \tau_{\eta}^*$. The condition that $\eta \in \Delta_{x,(2)}^Q$, means that $\tau_{\eta} h^2 \tau_{\eta} = h^2 \Leftrightarrow \tau_{\eta}^2 h = h$, thus $\tau_{\eta}^2 = \tau_{\eta}^{2*} = 1$. Thus, $\tau_{\eta} h \tau_{\eta} = h$. Conversely, $\tau_{\eta} h \tau_{\eta} = h$, means that $\tau_{\eta}^2 h^2 = h^2$, that should be compared with $\tau_{\eta} \tau_{\eta}^* h^2 = h^2$ and if $\operatorname{Im} \eta \neq 0$, we do not necessarily have the opposite inclusion.

Lemma 33.6. $\Delta_{x,(N)}^Q$ is decreasing as N increases.

Assume Γ is a circle-segment with center $\{0\}$ in a complex variable. Let $I_{\Gamma}(u_{\lambda}) = \int_{\Gamma} e^{\varphi} u_{\lambda} dx$. The Cousin-integral Φ can now be given as $\Phi = -\frac{\delta}{\delta y} I_{\Gamma}(u_{\lambda}) = \int_{\Gamma} (\frac{\delta}{\delta x} e^{\varphi}) u_{\lambda} dx$ and $\frac{\delta}{\delta y} I_{\Gamma}(u_{\lambda}) = -I_{\Gamma}(\frac{\delta}{\delta x} u_{\lambda})$, where y is assumed to lie in the same plane as x , thus $\Phi = \int_{\Gamma} e^{\varphi} \frac{\delta}{\delta x} u_{\lambda} dx$. We know that Φ can be analytically continued along transversals $\ni y$, such that $\operatorname{Im}(x - y) = 0$ for $x \in \Gamma$ (a real phase). $\tilde{\Phi}$ is constructed according to Cousin, by adding to Φ terms on the form $F_{\lambda} I_{\Gamma}(\frac{\delta}{\delta y} e^{-\varphi})/2\pi i = F_{\lambda} \log(b - y)$, where b is the one end point of Γ , given in local coordinates. Analogously, for the other (\pm) . $\tilde{\Phi}$ is then regular over the transversal, to \tilde{y} .

Assume now instead that $\operatorname{Re} h \sim_{\infty} h$ and $\eta = (x - y)$, with $\operatorname{Im} \eta \in \Delta(^t I_H)$. Given a disjoint decomposition of Γ , we now have $t\eta \in \Sigma_{x,y}(\operatorname{Re} h)$, $\forall t$ real. If we represent $\tilde{\Phi}$ as a I_{Γ} -integral for $\varphi = h(\psi)$, where ψ can be chosen as real, we have that for φ^k , it is sufficient to consider I_{Γ} with a real phase. In the case with a complex variable, we thus have $\Delta_{x,(N)}^Q \downarrow \{0\}$ as $N \uparrow \infty$. For the case with several variables, it follows immediately through the one-variable result, that $\{0\}$ is represented as an isolated point in $\Delta_{x,(N)}^Q$ for N sufficiently large.

We can now repeat the argument concerning the set of lineality $\Delta_{\mathbf{C}}$. Define the ideal $J = I(\Delta_{x,(1)}^Q)$. Let $V_1 = U_1 \setminus \Delta_{x,(1)}^Q \subset N(J) = U_1 = N(\mathcal{N}_{U_1})$. The

construction can be iterated, $J^2 = I(\Delta_{x,(2)}^Q)$, with $V_2 = U_2 \setminus \Delta_{x,(2)}^Q \subset U_2$. Since $\Delta_{x,(N)}^Q \downarrow \{0\}$, as $N \uparrow \infty$, since $\Delta_{x,(N)}^Q$ is defined by a homomorphism $(h: H \rightarrow H')$, we also have that V_N is decreasing as $N \uparrow \infty$. Thus, U_N is decreasing with respect to inclusion of sets, $0 \notin V_N$, for all N , but $\varphi \in J_0^N$ for all N . We have $\text{rad } J^N = \mathcal{N}_{U_N}$ for all N and $\varphi \in J_t^N$ means $t = 0$ and φ^N corresponds to a real phase, that is φ^N is reduced for weighted lineality.

For a generalization to the Cousin-integral in several variables, we can use the same argument. Thus, there is a positive N_0 , such that $\Delta_{x,(N)}^Q = 0$, $N \geq N_0$. Further, $\Delta_{x,(j)}^Q$ can, for any j , be realized by a locally injective C^∞ -mapping with a locally injective differential. The corresponding integral will take the form, for a positive constant C

$$\int_{\Gamma} \left(\frac{\delta}{\delta_{x_1}} \dots \frac{\delta}{\delta_{x_k}} e^{\lambda \varphi(x_1, \dots, x_k)} \right) u_{\lambda}(x_1, \dots, x_k, z) dx_1 \dots dx_k =$$

$$C \int_{\Gamma} e^{\lambda \varphi(x_1, \dots, x_k)} \frac{u_{\lambda}(x_1, \dots, x_k, z)}{(x_1 - y_1) \dots (x_k - y_k)} dx_1 \dots dx_k$$

34. THE PHASE IDEAL

The Hamilton-fields. Assume Ω a convex set, in the sense that any straight line between two points in Ω is contained in Ω . We assume that the trajectory we consider, γ , has an interior that is convex locally, so that in a neighborhood of a point on γ , in a small circle centered by a point, the inner of γ is situated on one side of the tangent to the point.

Consider the system

$$\frac{d\psi}{d\varphi} = P \quad \frac{dh(\psi)}{d\varphi} = Q$$

We then have $\frac{\delta\psi}{\delta\zeta_j} = P \frac{\delta\varphi}{\delta\zeta_j}$ and $\frac{\delta h(\psi)}{\delta\zeta_j} = Q \frac{\delta\varphi}{\delta\zeta_j}$ and we consider the Hamilton-fields $H_h = \sum_j \frac{dh}{d\varphi} \frac{\delta}{\delta\zeta_j} - \frac{\delta h}{\delta\zeta_j} \frac{d}{d\varphi}$. The Poisson-bracket can now be used to define an ideal over a domain Ω . Further, if Ω is a convex domain in the sense above and satisfies the two regularity conditions and if $h \equiv 0$ on Ω , then $e^{tH_h} \in \Omega$ for $|t| < t_0$

Assume h constant over ψ , that is $\psi \in (I_\mu)$, $h(\psi) = \mu\psi$ and $\Omega = N(I_\mu)$, we then have $\frac{dh(\psi)}{d\varphi} = 0$ and in any singular point to the system, $H_h(\psi) = 0$. In this case $\frac{d}{dt} e^{tH_h} = 0$ for $|t| < t_0$. If the order of zero of the point to P is infinite, $H_h(\psi) = 0$ in a ζ -neighborhood of the singular point. Consider $h(\psi) = \mu\psi$ and assume $\frac{\delta h(\psi)}{\delta\zeta_j} = \mu_j \frac{\delta\psi}{\delta\zeta_j} = \frac{Q}{P} \frac{\delta\psi}{\delta\zeta_j}$ and $\frac{d}{d\varphi} h = h \frac{d}{d\varphi}$. Then $\frac{\delta h}{\delta\zeta_j} \frac{d\psi}{d\varphi} = \frac{\delta Q}{\delta\zeta_j}$ and $\frac{dh}{d\varphi} \frac{\delta\psi}{\delta\zeta_j} = h(\frac{\delta P}{\delta\zeta_j})$. So, $H_h(\psi) = \sum_j h(\frac{\delta P}{\delta\zeta_j}) - \frac{\delta Q}{\delta\zeta_j}$. If we assume also $\frac{\delta Q}{\delta\zeta_j} = \mu_j \frac{\delta P}{\delta\zeta_j}$, we have $H_h(\psi) = 0$.

Assume h self-adjoint and such that h^2 is locally injective and consider $\eta(\psi)\psi = h(\psi)$. With these conditions η is a real-valued function and

$$\psi \frac{d\eta}{d\psi} = \frac{dh(\psi)}{d\psi} - \frac{h(\psi)}{\psi}$$

Assume further $\psi \frac{d\eta}{d\psi} = \mu - \frac{\overline{h(\psi)}}{\psi}$ for a real μ . Then, $\overline{\int_{|\psi|=1} h(\psi) d\psi} = \int_{|\psi|=1} \frac{d\eta}{d\psi} d\psi$. Further $\overline{\int_{|\psi|=1} \text{Im } h(\psi) d\psi} = 0$ because since h is locally defined through a polynomial, we have locally $h^* = \overline{h}$. Particularly, if we consider $e^{h(\varphi)} = \tilde{h}(e^\varphi)$ and consider the above relations with h replaced by \tilde{h}

$$\text{Im} \int_{|\psi|=1} e^{h(\varphi)} d\varphi = 0$$

The conclusion is that if the ideal that defines the phase is finitely generated, the imaginary part of the phase does not contribute to the corresponding operator.

34.1. Analytic continuation for I_Γ . Assume $g = g_0 + ic$, for a constant c and g_0 real. If there exist $g_j < g$ such that $\tilde{h}(u) - v \in H_{g_1}^{loc}$, we say that $\tilde{h}(u) \sim v$ in H_g^{loc} . In the same way $\int_\Gamma e^{\lambda h(g)} u dx \sim \int_\Gamma e^{\lambda g} v dx$. Assume $h_j(g) = h(g_j)$ and using continuity, $h_j(g) \rightarrow h(g)$ as $j \rightarrow \infty$, further $\int_\Gamma e^{\lambda g} \tilde{h}_j(u) dx \rightarrow \int_\Gamma e^{\lambda g} v dx$ in $(H_g^{loc})'$, modulo the equivalence above. Thus if we write $v = \tilde{h}(u)$ we have

$$\int_\Gamma e^{\lambda h(g_j - g)} u dx \rightarrow 0 \text{ as } j \rightarrow \infty$$

. This corresponds to analytic continuation for I_Γ over a real phase. Given lineality, we can chose g_j non-constant and real where g_j corresponds to \tilde{I}_Γ . Now consider

$$\tilde{I}_\Gamma - I_\Gamma = \int_\Gamma e^{\lambda g} (\tilde{h}_j(u) - \tilde{h}(u)) dx = \int_\Gamma e^{\lambda (h_j(g) - h(g))} u dx$$

and if $h_j(g) - h(g) < \varphi$ we get for $\tilde{I}_\Gamma \sim I_\Gamma$, a remainder term that gives an exponentially small contribution to the integral representation. Trivially we have if $f \sim g$ in $H_\varphi^{loc}(\Omega)$ that there exists $g_0 \in H(\Omega)$ such that $g \sim g_0$ and $f \sim_m g_0$.

Further there exist I_Γ^0 such that $\tilde{I}_\Gamma - I_\Gamma^0 \sim_m 0$.

Assume as before $\eta \in \Sigma_{x,y}$ implies $T_\eta h = h$ for $T_\eta h = \tau_\eta h \tau_\eta$. Consider $\Sigma_0 = \{x < tx, \eta > \leq 0 \quad \forall t\}$ or equivalently $\{x < x, t\eta > \leq 0 \quad \forall t\}$. Let $\tilde{F}_\eta = T_\eta - I$, where F_η is considered in $H'(\Sigma_0)$. Further, $\Sigma^0 = \{\eta \mid T_\eta - I \text{ of type } 0\}$. We see that if $\eta \in \Sigma_{x,y}$, then Σ_0 contains entire lines and we can in this way define a conic neighborhood of $\Sigma_{x,y}$ according to Σ^0 .

We have noted that $\Delta_x^Q \downarrow \{0\}$ as $N \uparrow \infty$ and we assume h is algebraic over $I(\Delta_x^Q) = \{\varphi \mid \varphi = 0 \text{ on } \Delta_x^Q\}$. If h^N is locally injective, we have also that g is real. Let otherwise L be such that $h(L) = I(\Delta_x^Q)$, then $g^N \in L$ and $g \in \text{rad}(L)$. Obviously h^N is locally injective over $I(\Delta_x^Q)$, for large iteration indexes, which means that the corresponding g defines a real phase. More precisely, define (L_1) according to $h(L_1) = I(\Delta_1)$ and (L_2) according to $h^2(L_2) = I(\Delta_2)$ and so on. The proposition is thus that if $(I) = (I)(0)$ is defined through $(h, h^2, h^3, \dots, h^{N-1})$ according to a chain of ideals $L_1 \subset \dots \subset L_{N-1} = (I)$, then h^{N+1} is locally injective over (I) .

If $\varphi \in (I)$ is such that $\varphi = \sum_j c_j h^j(g_j)$ for $g_j \in L_j$ and since we are dealing with geometric ideals, we can make the decomposition disjoint such that $\varphi = h(\sum_j d_j g_j) \exists g_j \in J_j$ and $(I) = \oplus_j J_j$. If $\varphi \in I(\Delta_1)$ then $h(\varphi) = h(g^2)$ for

$g \in (L_2)$. We now have that $N(h(I(\Delta_1))) = N(h^2(L_2)) = \Delta_2 \subset \Delta_1$. Thus, $h(\varphi) = 0$ implies $\varphi = 0$ and we have also that h is locally injective on $(I)(\Delta_1)$.

Note finally that if $R = \{\varphi \mid \varphi \text{ real}\}$, we have $h: R \rightarrow R$ and $\text{rad}(R) \rightarrow \text{rad}(R)$ and further $\text{rad}(L_1) \subset h(L_2)$. In the more general case, h is algebraic in the tangent-space and singular points are points for which g is constant. These constant phases can thus be approximated by real and non-constant ones.

34.2. The quantitative version of Rückert's theorem. Assume the ideal (I) defined over a domain of holomorphy Ω such that for all $\varphi \in (I)$, $h(\varphi)$ is in the kernel of an elliptic operator. We then have that $e^{h(\varphi)}$ is a polynomial over $\Omega \setminus \mathcal{R}$. Particularly, if $\tilde{h} = h$ we have that the polar set to $h(\varphi)$ is an algebraic set. Assume $h(e^\varphi) = P/Q$ for P, Q polynomials and $Q^* = Q$ hypoelliptic. We are going to prove that $h(e^\varphi) = V + \lambda Q$, for a polynomial V in for instance x_1 and we make the approach $\lambda Q = \tilde{h}(e^\varphi)$ and

$$\lambda = \int_{|\xi|=r'} e^{h(\varphi)} \frac{1}{Q} d\xi$$

If the representation can be proven in the vicinity of the real infinity, then this implies assuming $P \prec Q$ that $e^{h(\varphi)} \prec Q$. Further $\delta\lambda \prec \prec I$ given that $\delta P \prec \prec Q$. We will use a special case of a Lemma by H. Cartan (cf. [4]). Let $\Delta_r = \{(x_1, \dots, x_r) \mid |x_j| \leq r_j \ \forall j\}$ and $\Delta'_r = \{(x_2, \dots, x_r) \mid |x_j| \leq r_j \ \forall j\}$ be polycylinders in the complex space. If Q has zero's locally in Δ_r and if f is holomorphic on Δ_r then there are λ, V such that $f = \lambda Q + V$ where the polynomial in x_1 has degree $\deg Q - 1$. If $|f| \leq 1$ on Δ_r we have that the coefficients for V are holomorphic on Δ'_r and of modulus $\leq M$ for a positive constant M . Further $|\lambda| \leq M$ on Δ_r . The definition of λ can be proven independent of r' close to r . We will briefly sketch the proof. If v_p are the coefficients to the polynomial V we have that $|v_p| \leq c'$ on $\Delta_{r'}$ for a positive constant c' . Thus there exists a positive constant c'' such that $|\lambda Q| \leq c''$ on Δ_r . For $(x_2, \dots, x_n) \in \Delta'_{r'}$ and $|x_1| = r_1$ we have that $|Q| > c$ for a positive constant c why using the maximum-principle we have that $|\lambda| \leq M$ on Δ_r for a positive M .

Consider again the polar set $(I) = \{\varphi \mid e^\varphi \rightarrow 0 \ x \rightarrow a\}$ and its closure with respect to Whitney-norm $\overline{(I)}$. Thus $\varphi \in \overline{(I)} \Leftrightarrow \varphi \rightarrow 0 \ (\varphi \rightarrow \text{const.}) \ x \rightarrow a$ and we assume also that the limit is uniform in a neighborhood of the point a . We can now define a module of uniform approximation of 1 (or of a constant) through non-constant functions $\psi = e^\varphi$ with φ real. Denote this module \mathcal{M}_a . It is evident in the minimally defined situation that $\widehat{\mathcal{M}}_a \subset J_h$, for a homomorphism h and that $\overline{\mathcal{M}}_a = \mathcal{M}_a$ (Whitney's theorem).

Assume Ω a domain of holomorphy and $\Gamma = \text{bd } \Omega$ a compact set, further that f_1, \dots, f_p are holomorphic functions on $\overline{\Omega}$, constituting a pseudo-base for a geometric ideal (I) . We know that there is a module of coefficients, $c_j \in \mathcal{M}$, holomorphic on a compact polycylinder $\Delta \subset \Omega$ such that $\sum_j c_j f_j = 0$ on Δ . We also assume the equation $\sum_j d_j f_j = 1$ solvable on Γ . If w is an algebraic homomorphism $(I) \rightarrow (I)$ such that $w(f_j) = \text{const} f_j$ for all j , then we obviously have $\mathcal{N} = w^{-1}\mathcal{M}$ is a module such that we can solve the equation $\sum_j a_j f_j \in J_w$,

where $(J_w) = \ker w$. If w is reduced and \mathcal{M}_a is the module for uniform approximation of 1, that is $c_j \rightarrow c_j^0$ such that $\sum_j c_j^0 f_j \rightarrow 1$ on Γ , then the approximating functions can be selected from \mathcal{N} . If we instead consider a homomorphism $w:(I) \rightarrow (I)$ as factorized through $\tau_\epsilon g$ for an algebraic homomorphism g , we can form \mathcal{N} as before and solve the homogeneous equation modulo monotropy. For the module of uniform approximation, we can consider the equations $dw(\sum_j a_j f_j) = 0$ on Γ and $w(\sum_j a_j f_j) = 0$ on Ω . If w is such that w^2 is locally injective we have the minimally defined situation and we can always find $g \in (J_w)(\Omega)$ such that $g \rightarrow g^0$ uniformly in a neighborhood of Γ . In the case where w is an algebraic homomorphism, it is sufficient to chose $\mathcal{N} = (J_w)(\Omega)$ and in the case of a factorized homomorphism we have the same proposition modulo monotropy.

We have earlier considered the example $\Omega = \Sigma$ and $\Gamma = \Delta_x^Q$ and the ideal $(J) = \{g \mid dh(g) = 0\}$ for a homomorphism w such that dw is reduced (locally injective) and $h(g) = w(\hat{g}) - \hat{w}(g)$. We write $a_j = e^{\varphi_j}$ and assume that $w(a_j) = \hat{w}(\varphi_j) \rightarrow 0$ as $x \rightarrow a$ for $x \in V$, V a neighborhood of a . If U is the dual to V in Exp , we see that $w(\varphi_j) \in \mathcal{E}'^{(0)}(U)$. The problem is now to determine the support for these measures. We can also assume $|\hat{w}(\varphi_j) - Q_j(\frac{1}{x-a})| < \epsilon$ for $x \neq a$, x close to a . This means that the phase $w(\varphi_j)$ can be seen locally as a polynomial operator $Q_j(D)\delta_A$, where A corresponds to a through the duality. Under the condition $Df \prec\prec I$, it is sufficient to consider operators Q of first order.

A comparison with Cartan's version of Rückert's Nullstellensatz gives with the same approach as earlier that if $h(e^\varphi) = 0$ we have modulo monotropy that the condition $|h(e^\varphi)| \leq \epsilon$ gives $|\lambda| \leq M\epsilon$ for a positive M and a small positive ϵ . If $(I) = \{g \mid e^g \rightarrow 0 \mid x \rightarrow a\}$ and $g = h(\varphi)$ we can form $(I)_m = \{h(\varphi) \mid |h(e^\varphi)| < \epsilon\} = \{h(\varphi) \mid |\hat{h}(\varphi)| < \epsilon\}$. We can now use Paley-Wiener's theorem modulo monotropy and we see that $h(\varphi) \in \mathcal{E}'^{(0)}$, thus $(I)_m$ can be realized through measures. [19]

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